JOURNAL OF FUNCTIONAL ANALYSIS **153**, 276–319 (1998)
ARTICLE NO. FU973181

Uniqueness of Schrödinger Operators Restricted in a Domain

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Received January 31, 1997; accepted July 23, 1997

Let $S := -\Delta/2 + V$ be the Schrödinger's operator defined on $C_0^\infty(D)$ where D is a (open) domain in $\mathbf{R}^d$. By means of the asymptotic behavior of V near the boundary ∂D, we give the necessary and sufficient conditions to the essential Markovian self-adjointness of S for the nonnegative potential V, and to the uniqueness of S in $L^1(D)$ for general V.

Key Words: L^1-essential generator; essential Markovian self-adjointness; Schrödinger operators; Dirichlet forms.

0. A NAIVE PROBABILISTIC INTERPRETATION OF THE UNIQUENESS

Let D be an (open) domain in $\mathbf{R}^d$ with its boundary ∂D. We denote by $C_0^\infty(D)$ the space of all infinitely differentiable real functions on D with compact support. Consider the Schrödinger's operator $S := (-\Delta/2 + V, \mathscr{D}(S)$ (the domain) $= C_0^\infty(D))$ where $V: D \to \mathbf{R}$ is a Borel measurable potential. The essential self-adjointness (in abridge: *e.s.a.*) of S in $L^2(D, dx)$, equivalent to the uniqueness of the self-adjoint extension of S or the unique solvability of Schrödinger's equation in $L^2(D)$, has been studied extensively (see Kato [Ka], Reed and Simon [RS], Simon [Si] etc. for survey), because of its importance in Quantum Mechanics. Very general results are known when D is the whole space $\mathbf{R}^d$. In the case where D is a strict sub-domain, sharp results are known only when $d = 1$ by means of the Weyl's *limit point-limit cycle* criterion (see [RS, p. 146–161]). This theory can be applied to the multidimensional case only in some special situations ([RS, X.4], [Si] e.g.).

Assume for a moment V is locally square integrable and nonnegative in D. It is well known that the e.s.a. of S is determined by the boundary behaviors (and in Physics different boundary behaviors correspond to different physics).

0022-1236/98 $25.00

We interpret now the uniqueness of the *reasonable* extension of S in a very naive (probabilistic) way. It is well known that $\Delta/2$ generates a (free) Brownian motion $(B_t)_{t\geqslant 0}$ in $\mathbf{R}^d$ defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, (\mathbf{P}_x)_{x\in\mathbf{R}^d})$ $(\mathbf{P}_x(B_0=x)=1)$. One can prove rather easily that the killed Feynman–Kac semigroup

$$P_t^{D,V}f(x) := \mathbf{E}^x 1_{[t<\tau_D]} f(B_t)\cdot \exp\left(-\int_0^t V(B_s)\,ds\right) \tag{0.1}$$

is a strongly continuous semigroup of bounded operators (in abridge: C_0-semigroup) on $L^p(D)$, $\forall p\in[0,+\infty)$, and its generator $A_D^{(p)}$ is an extension of S: $C_0^\infty(D)\to L^p(D)$ for $1\leqslant p\leqslant 2$ (see Section 2), where $\tau_D := \inf\{t>0; B_t\notin D\}$ is the first exiting time of D.

To have the uniqueness, intuitively the repulsive potential $V=V^+$ should grow rapidly to infinity near ∂D so that

(I) *the particle can not reach* ∂D, *this means by* (0.1) *that*

$$\mathbf{P}_x\left(\int_0^{\tau_D} V^+(B_s)\,ds + \tau_D = +\infty\right) = 1 \qquad \text{for a.e.} \quad x\in D; \tag{C1}$$

or

(II) *the particle could touch* ∂D *but cannot survive after that moment, this means for* dx-*a.e.* $x\in D$,

$$\mathbf{P}_x\left(\int_0^{\tau_D+\varepsilon} (1+V(B_t^R))\,ds = +\infty, \forall\varepsilon>0\right) = 1 \tag{C2}$$

where (B_t^R) *is the reflecting Brownian motion* (*in abridge*: *RBM*).

From (0.1), these two conditions say that the killed Feynman–Kac semigroup should be the unique one generated by S *in some sense*. However, the above intuitive pictures do not lead to the essential self-adjointness of S unfortunately. One typical example is given by

$$D=\mathbf{R}\backslash\{0\}, \qquad V(x)=\beta/x^2 \qquad (\beta>0).$$

V satisfies (C1) but S is essentially self-adjoint if and only if $\beta\geqslant 3/8$ (see [RS, Th. X.10]).

The main purpose of this paper is to find the senses in which the intuitive pictures above guarantee the uniqueness of the extensions of S and to find analytical descriptions of (C1) and (C2). We shall show that (C1) implies the uniqueness of the C_0-semigroups in L^1 generated by S, but (C2) characterizes the uniqueness of the **sub-Markov** semigroups generated by S, called often the essential Markovian self-adjointness (in abridge: *e.m.s.a.*) in the current works.

This paper is organized as follows. The main results and related works are presented in the next Section. Some preliminaries and lemmas are given in Section 2. Their proofs are given in Section 3, 4, 5. Some applications to the unique solvability of partial differential equations are furnished in Section 6, where the reader finds the other motivation of this work. Finally several examples are presented in Section 7.

1. MAIN RESULTS

1.1. *Uniqueness of the Schrödinger Operators in $L^1(D)$*

The main purpose of this paper is to prove that the intuitive pictures above hold in somewhat weaker sense (than the e.s.a.). Throughout this paper we assume that

$$V^+ \in L^1_{loc}(D) \text{ and } V^- \text{ belongs to the Kato class in } \mathbf{R}^d \tag{H1}$$

where

$$L^p_{loc}(D) = \{f\colon D \to \mathbf{R} \text{ measurable: } f1_{\mathbf{K}} \in L^p(D, dx), \forall \mathbf{K} \text{ compact} \subset D\}$$

$V^-(x) = 0$, $\forall x \notin D$ (any function f on D will be regarded as a function on $\mathbf{R}^d$ in this way), and the Kato class is given by (see e.g. [Si]),

Definition. A Borel function v is called in Kato class $\mathscr{K}$, iff

$$\lim_{\delta \downarrow 0} \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant \delta} |g(x-y)\, v(y)|\, dy = 0, \tag{1.1}$$

where

$$g(x) = |x|^{2-d} \quad \text{if } d \geqslant 3; \qquad \ln \frac{1}{|x|} \quad \text{if } d = 2; \qquad 1 \quad \text{if } d = 1.$$

Let $\mathscr{K}_{loc}(D) = \{V;\ V1_K \in \mathscr{K},\ \forall \text{ compact } K \subset D\}$. Introduce

$$P^D_t f(x) := \mathbf{E}^x f(B_t)\, 1_{[t < \tau_D]} \text{ (the killed BM semigroup)}$$
$$p^D_t(x_0, x) = P^D_t(x_0, dx)/dx \text{ (the density)}$$
$$G_D(x_0, x) := \int_0^{+\infty} p^D_t(x_0, x)\, dt \in [0, +\infty],$$

(the last is the Green function w.r.t. $(\Delta/2, D)$). We need also

DEFINITION. We say that a Borel subset O of D is charged by the BM until τ_D, if $\forall x \in D$, there is a $\mathscr{F}_{\tau_D}$-random time τ such that $\tau < \tau_D$, $\mathbf{P}_x$-a.s. and

$$\mathbf{P}_x(B_t \in O; \forall t \in (\tau, \tau_D)) > 0$$

In the above definition, we can substitute $\forall x \in D$ by $\exists x_0 \in D$ if D is connected (left to the reader). We can now state

THEOREM 1.1. *Assume* (H1). *Then the killed Feynman–Kac semigroup defined in* (0.1) *is a* C_0*-semigroup on* $L^1(D)$, *and its generator* $-A_D^{(1)}$ *is an extension of* $-S$: $C_0^\infty(D) \to L^1(D)$. *And we have the equivalence between*

(i) *the closure of* $-S$ *in* $L^1(D)$ *is the generator of a* C_0*-semigroup on* $L^1(D)$ (*say,* $-S$ (*or* S *with some abuse*) *is an* **essential generator** *in* L^1, *or* L^1*-e.gr. in abridge*);

(ii) *the closure of* S *in* $L^1(D)$ *is* $A_D^{(1)}$;

(iii) V^+ *satisfies the condition* (C1).

In this case $(P_t^{D,V})_{t \geqslant 0}$ *is the unique strongly continuous semigroup of bounded operators on* $L^1(D)$ *such that its generator is an extension of* S.

Moreover if $V^+ \in \mathscr{K}_{loc}(D)$ *and* D *is connected,* (C1) *is equivalent to*

(iv) $\exists$ (*or* $\forall$) $x_0 \in D$ *such that for every open* $O \subset D$, *charged by the* BM *until* τ_D,

$$\int_O G_D(x_0, x)(1 + V^+)(x)\,dx$$

$$= \mathbf{E}^{x_0} \int_0^{\tau_D} 1_O(B_t)(1 + V^+)(B_t)\,dt = +\infty. \tag{1.2}$$

Remarks. (1.i) That $V \in L^1_{loc}(D)$ in (H1) is a minimal condition for $S(C_0^\infty(D)) \subset L^1(D)$.

(1.ii) Recall that if moreover $V \in L^2_{loc}(D)$, S is *e.s.a.* iff the closure of $-S$ in $L^2(D)$ is the generator of a strongly continuous semigroup of symmetric bounded operators in $L^2(D)$. So *the* L^1*-e.gr. property of* S is exactly the counterpart of the *e.s.a.* in L^1.

(1.iii) Let $D = \mathbf{R}^d \setminus N$ where N is of zero capacity-(1,2) (or it is dx-polar for the BM) and $V \in L^1_{loc}(D)$ satisfy (H1). As (C1) is satisfied, then S is L^1-e.gr. by Theorem 1.1. When $N = \emptyset$, this is the counterpart of Kato's well known theorem (see [Si]): S is e.s.a. once $V^+ \in L^2_{loc}(\mathbf{R}^d)$.

(1.iv) If the killed Brownian motion is recurrent (i.e., $G_D \equiv +\infty$, this happens iff ($d=1$ and $D=\mathbf{R}$) or ($d=2$ and $Cap(D^c)=0$)), the property (1.2) holds always.

Regard now more closely the semi-analytic semi-probabilistic condition (1.2).

If D is Lipchizian (see Remark (v) below), obviously every open ball $B(z, r)$ where $z \in \partial D$, $r>0$ charged by the BM until τ_D. Then if S is L^1-e.gr., necessarily we have

$$\int_{B(z,r)} G_D(x_0, x)\, V^+(x)\, dx = +\infty, \qquad \forall z \in \partial D, \quad r>0 \tag{1.3}$$

i.e., $G_D(x_0, x)\, V^+(x)$ is not in L^1_{loc} near any point of ∂D. But unfortunately this very explicite necessary condition is not sufficient, see Section 7 for one such counter-example.

The following result says that a stronger but simpler form (purely analytical) of (1.3) is sufficient.

PROPOSITION 1.2. *Assume that D is a C^2 domain, $V \geqslant 0$. Let $\rho(x)$ be the distance of x with ∂D.*

(a) *If $\forall z_0 \in \partial D$, $\forall r_0 > 0$ small,*

$$\int_0^{r_0} r \tilde{V}(r)\, dr = +\infty \tag{1.4}$$

where $\tilde{V}(r)$ is the essential infimum of V restricted to

$$\Gamma_r := [\rho(\cdot) = r] \cap B(z_0, r_0) \cap D$$

w.r.t. the surface area measure σ_r of Γ_r, then V satisfies (C1).

(b) *If $\forall z_0, r_0$ as in* (a), *there is a constant $C \geqslant 1$ such that*

$$V(x) \leqslant C \tilde{V}(\rho(x)),\ dx\text{-a.e.} \qquad \text{on} \quad B(z_0, r_0) \cap D$$

then (1.4) *is also necessary for* (C1).

Remarks. (1.v) A C^2-domain means that $\forall z \in \partial D$, there is $r>0$ such that

$$B(z, r) \cap D = B(z, r) \cap \{(x_i) : x_d > \phi(x_1, ..., x_{d-1})\}$$

where (x_i) is a system of local coordinates on $B(z, r)$, and $\phi \in C^2$. If ϕ is only Lipchizian, D is called a Lipchizian domain.

For a bounded C^2-domain D in $\mathbf{R}^d$ with $d \geqslant 2$, it is well known that (due to Gilbarg and Trudinger, see [CZ, p. 143]) $(\partial/\partial n_z)\, G_D(x_0, z) > 0$ exists and continuous on $(x_0, z) \in D \times \partial D$, where n_z is the inner normal vector. Therefore as $x \to z$ along the normal direction,

$$G(x_0, x)/\rho(x) \to \frac{\partial}{\partial n_z} G_D(x_0, z) > 0$$

Hence (1.4) is stronger than (1.3), and they are equivalent in the situation of Proposition 1.2.b.

Our proof of this theorem, using only simple stochastic calculus, does not rely on this savant property.

1.2. *The Essential Markovian Self-Adjointness*

We work now in the following context (more restrictive than (H1)):

$$V \geqslant 0 \qquad \text{and} \qquad V \in L^2_{loc}(D) \tag{H2}$$

under which S: $C_0^\infty(D) \to L^2(D)$ is a symmetric operator. We study now another property of S: the essential Markovian self-adjointness (in abridge: *e.m.s.a.*). This notion attracts much attention in recent researches, see Albeverio and Kusuoka [AK], Röckner and Zhang [RZ1, 2], Song [So], Takeda [Ta1, 2], [FOT] etc., about $\Delta + (\nabla\phi/\phi)\nabla$, that they called generalized Schrödinger operator.

We present this notion in the actual context. Let

$$\mathscr{A}_+(S) = \{A \text{ is a definitely nonnegative self-adjoint extension of } S\} \tag{1.5a}$$

$$\mathscr{A}_M(S) := \{A \in \mathscr{A}_+(S);\ T_t^A \text{ is sub-Markov}\}. \tag{1.5b}$$

where $(T_t^A := \exp(-tA))_{t \geqslant 0}$ is the semigroup generated by the self-adjoint operator $-A$. The *essentially Markovian self-adjointness* (*e.m.s.a.*) of S means that $\mathscr{A}_M(S)$ is a **singleton**.

Since a symmetric sub-Markovian C_0-semigroup in $L^2(D)$ is also a C_0-semigroup in $L^1(D)$, the e.m.s.a. of S is then weaker than the L^1-e.gr. in Theorem 1.1. We get so

Corollary 1.3. *Assume* (H2). *If V satisfies* (C1), *then S is e.m.s.a.*

Whether (C1) *is necessary to the e.m.s.a. of S?*

The answer will be NO. To present precisely our answer, we need some more languages. Let $A_1, A_2 \in \mathscr{A}_+(S)$, we say that A_1 is larger than A_2 (in the form domain sense), if

$$\mathscr{D}(\mathscr{E}_{A_1}) \supset \mathscr{D}(\mathscr{E}_{A_2}) \qquad \text{and} \qquad \mathscr{E}_{A_1}(f, f) \leqslant \mathscr{E}_{A_2}(f, f), \qquad \forall f \in \mathscr{D}(A_2) \tag{1.6a}$$

where

$$\mathscr{E}_A(f, g) := \langle \sqrt{A}\, f, \sqrt{A}\, g \rangle_{L^2(D)}, \qquad \forall f, g \in \mathscr{D}(\mathscr{E}_A) = \mathscr{D}(\sqrt{A}) \tag{1.6b}$$

is the quadratic form associated to a nonnegative self-adjoint operator A (when $A \in \mathscr{A}_M(S)$, $\mathscr{E}_A$ is called its associated Dirichlet form).

Let

$$H^{1,2}(D) = \{ f \in L^2(D); \nabla f \in L^2(D) \} \tag{1.7a}$$

where ∇f is taken in the Schwarz distributions space $\mathscr{D}'(D)$, and

$$H_0^{1,2}(D) = \text{the closure of } C_0^\infty(D)\ H^{1,2}(D) \tag{1.7b}$$

THEOREM 1.4. *Assume* (H2).

(a) *The minimal element in* $\mathscr{A}_M(S)$ *w.r.t. the order* (1.6*a*) *is the s.a. operator* A_D *associated to the Dirichlet form*

$$\begin{aligned} \mathscr{E}_D^V(f, g) &= \tfrac{1}{2} \int_D \nabla f \cdot \nabla g\, dx + \int_D V f g\, dx, \\ \forall f, g &\in \mathscr{D}(\mathscr{E}_D^V) = H_0^{1,2}(D) \cap L^2(V\, dx) \end{aligned} \tag{1.8}$$

And the maximal element in $\mathscr{A}_M(S)$ *w.r.t. the order* (1.6*a*) *is the s.a. operator* A_R^V *associated to the Dirichlet form*

$$\begin{aligned} \mathscr{E}_R^V(f, g) &= \tfrac{1}{2} \int_D \nabla f \cdot \nabla g\, dx + \int_D V f g\, dx, \\ \forall f, g &\in \mathscr{D}(\mathscr{E}_R^V) = H^{1,2}(D) \cap L^2(V\, dx) \end{aligned} \tag{1.9}$$

In particular S is e.m.s.a. iff $\mathscr{D}(\mathscr{E}_D^V) = \mathscr{D}(\mathscr{E}_R^V)$.

(b) *Assume moreover that ∂D is Lipchizian, whose surface area measure is denoted by $\sigma(dz)$. The following properties are equivalent*:

(b.i) *S is e.m.s.a.*;

(b.ii) $C(\bar{D}) \cap \mathscr{D}(\mathscr{E}_R^V)$ ($C(\bar{D})$ is the space of continuous functions on $\bar{D}$) *is a form core of* $\mathscr{E}_R^V$ *and*,

$$\forall z \in \partial D, \forall r > 0: \qquad \int_{B(z,r)} V\, dx = +\infty \tag{1.10}$$

(b.iii) *For σ-a.e. $z \in \partial D$,*

$$\mathbf{P}_z\left(\int_0^{\varepsilon} V(B_t^R)\, ds = +\infty, \forall \varepsilon > 0\right) = 1 \tag{1.11}$$

where $((B_t^R), \mathbf{P}_z)$ is the reflecting (at ∂D) Brownian motion issued of $z \in \bar{D}$.

(b.iv) *V satisfies* (C2);

(b.v) *For every finely open or B^R-quasi open $O \subset \bar{D}$ w.r.t. the reflecting BM B^R such that $\sigma(O \cap \partial D) > 0$,*

$$\int_O V\, dx = +\infty \tag{1.12}$$

Remarks. (1.vi) We refer to Bass and Hsu [BH1, 2] and Fukushima and Tomisaki [FT] for the construction of the reflecting Brownian motion (in abridge: RBM). Since the sub-Markov semigroup modelizes the heat diffusion, so the notion of *e.m.s.a.* is purely of *particle feature*.

(1.vii) The two analytical conditions (b.ii), (b.v) present their advantage and disadvantage. In (b.ii), the second property, (1.10), is very explicite, but the first of (b.ii), meaning roughly that $\mathscr{E}_R^V$ is regular, is a current assumption, but difficult to check (See Albeverio and Ma [AM2] for the studies of nonregular Dirichlet form and the references). In Section 7, we shall construct a counter-example showing that (1.10) alone is not sufficient to (b.i).

About (b.v), it is trivially stronger than (1.10). We refer to Sturm [St] and Getoor [Ge] for the general studies of (1.11).

Here is a counterpart of Proposition 1.2 for the e.m.s.a.

PROPOSITION 1.5. *Assume that D is a C^3 domain. If*

$$\forall z_0 \in \partial D, \forall r_0 > 0 \text{ small: } \int_0^{r_0} \tilde{V}(r)\, dr = +\infty, \tag{1.13}$$

where $\tilde{V}(r)$ is given in Proposition 1.2.a, *then S is e.m.s.a.. Inversely if V satisfies the condition in Proposition* 1.2.b, *the condition* (1.13) *is also necessary to the e.m.s.a. of S.*

Remarks. (1.viii) The most intuitive conditions are (C2) and (1.11). The condition (C1) for the L^1-e.gr. in Theorem 1.1. means that ∂D is S-repulsive (i.e., the particle can not reach ∂D), but (C2) means intuitively that ∂D is *S-hard* (*i.e.*, the particule could reach ∂D, but could not survive *once reached*).

(C1) is obviously stronger than (C2). One asks naturally whether (C1) is strictly stronger than (C2). At first glance, they are very close. But their difference can be seen clearly from Theorem 1.1(iv) and Theorem 1.4(b.v), and still more explicite in one-dimensional case:

COROLLARY 1.6. *Let* $d=1$, *and* $D=(a,b)$ *where* $-\infty\leqslant a<b\leqslant+\infty$.

(a) *Assume* (H1). (C1) *holds* ($\Leftrightarrow$ *the* L^1-*e.gr. of* S), *iff* $\forall z=a,b$ *finite*, $\forall r>0$ *small*,

$$\int_{B(z,r)} V(y)\,|y-z|\,dy=+\infty \tag{1.14}$$

(b) *Assume* $0\leqslant V\in L^2_{loc}(D)$. (C2) *holds* ($\Leftrightarrow$ *the e.m.s.a. of* S), *iff* $\forall z=a,b$ *finite*, $\forall r>0$ *small*

$$\int_{B(z,r)} V(y)\,dy=+\infty \tag{1.15}$$

This result follows directly from Theorem 1.1 and Theorem 1.4, because a set charged by the BM until τ_D in Theorem 1.1(iv) or a finely open set in Theorem 1.4(b.v) contain $B(z,r)\cap D$ for r small. (Of course it follows still more directly from Proposition 1.2 and 1.5, but in reality we shall prove Proposition 1.2 and 1.5 from Corollary 1.6).

Especially for $D=(0,+\infty)$, $V(x)=\beta/x^\alpha$, where $\beta>0$, $\alpha\in\mathbf{R}$, S is L^1-e.gr. iff $\alpha\geqslant2$, by (1.14). However S is e.m.s.a. iff $\alpha\geqslant1$, by (1.15). We have said at the beginning that for $\alpha=2$, S is e.s.a. iff $\beta\geqslant3/8$. This seems strange to one probabiliste (I still have not well understood this phenomena physically, but I suggest the following: the self-adjointness of S should be interpreted from the viewpoint of *particle-wave* as in Schrödinger's equation, but the intuitive pictures (C1) and (C2) are only of *particle feature*.)

2. SEVERAL LEMMAS

2.1. *Some Properties of the Killed Feynman–Kac Semigroup*

We continue to employ the notations in the previous Sections. Recall at first several well known properties of the Kato class $\mathcal{K}$ (see [Si] for an excellent survey):

(i) $L^p_{loc}(D) \subset \mathscr{K}_{loc}(D) \subset L^1_{loc}(D)$ if $d \geqslant 2$ and $p > d/2$ or if $d = p = 1$.

(ii) (due do Aizenman and Simon [AS]): a measurable function $v \in \mathscr{K}$ (over $\mathbf{R}^d$) iff $|v| \in \mathscr{K}$, iff the Feynman-Kac semigroup

$$P_t^v f(x) := \mathbf{E}^x f(B_t) \cdot \exp\left(-\int_0^t v(B_s)\, ds\right), \tag{2.1}$$

is bounded in all $L^p(\mathbf{R}^d, dx)$, $p \in [1, +\infty]$, strongly continuous for $p \in [1, +\infty)$, and

$$\lim_{t \to 0^+} \sup_{x \in \mathbf{R}^d} \mathbf{E}^x \exp\left(\int_0^t |v|\,(B_s)\, ds\right) = 1. \tag{2.2}$$

In particular,

$$\int_0^\cdot |v|\,(B_s)\, ds < +\infty \text{ over } \mathbf{R}^+, \mathbf{P}_x\text{-a.s.}, \qquad \forall x \in \mathbf{R}^d \tag{2.3}$$

(iii) for $v \in \mathscr{K}$, the bilinear form

$$q_v(f, g) := \int_{\mathbf{R}^d} vfg\, dx, \qquad \mathscr{D}(q_v) := L^2(\mathbf{R}^d, |v|\, dx)$$

is bounded by the canonical Dirichlet form

$$\mathscr{E}(f, g) := \tfrac{1}{2} \int_{\mathbf{R}^d} \nabla f \cdot \nabla g\, dx, \qquad \mathscr{D}(\mathscr{E}) = H^{1,2}(\mathbf{R}^d), \tag{2.4}$$

with an arbitrary small bound ε. And the multiplication operator V is bounded by Δ with an arbitrary small constant ε on $L^1(D)$.

(iv) For $v \in \mathscr{K}$, $\forall t > 0$, $\|P_t^v\|_{p, \infty} < +\infty$ for all $p \in [1, +\infty]$.

Note that (H1) guarantee

$$\int_0^t V^+(B_s)\, ds < +\infty, \qquad \mathbf{P}_x\text{-a.s. on } [t < \tau_D], \quad \forall a.e.\ x \in D \tag{2.5a}$$

$$\int_0^t V^-(B_s)\, ds < +\infty, \qquad \forall t \geqslant 0, \quad \mathbf{P}_x\text{-a.s.}, \quad \forall x \in \mathbf{R}^d \tag{2.5b}$$

where (2.5a) follows from Fubini's theorem, (2.5b) follows from (2.3) (attention: (2.5a) still holds for quasi-every $x \in D$, but may fail for all $x \in D$, see Sturm [St]). By Fatou's lemma, $\int_0^\cdot V(B_s)\, ds$ is an *left-continuous* additive functional with values in $(-\infty, +\infty]$.

We begin by studying the killed Feynman–Kac semigroup (0.1). Recall that the extensive studies are paid mainly to the case where V^+ belongs also to the Kato class (see Aizenman and Simon [AS], Chung and Zhao [CZ] and the references therein). The following lemma is starting point for us, and it is a special case of [AM] and [FOT, Chap. VI] when $V = V^+ \geqslant 0$.

LEMMA 2.1. (a) *$(P_t^{D,V})_{t \geqslant 0}$ is a C_0-semigroup on $L^p(D, m)$ for every $p \in [1, +\infty)$, and it is a semigroup of bounded operators in $L^\infty(D, m)$, where m denotes the Lebesgue measure.*

(b) *$P_t^{D,V}$, $t \geqslant 0$ are self-adjoint in $L^2(D, m)$. Let $(\mathscr{E}_D^V, \mathscr{D}(\mathscr{E}_D^V))$ be the Dirichlet form associated to this strongly continuous symmetric semigroup, then*

$$\begin{aligned} \mathscr{D}(\mathscr{E}_D^V) &= H_0^{1,2}(D) \cap L^2(V^+ m), \\ \mathscr{E}_D^V(f, g) &= \tfrac{1}{2} \int_D \nabla f \cdot \nabla g \, dx + \int_D fgV \, dx \end{aligned} \tag{2.6}$$

(c) *$C_0^\infty(D)$ is a form core of $\mathscr{E}_D^V$.*

Proof. (a) For every $f \in L^p(D, m)$, we have for every $x \in D$,

$$|P_t^{D,V} f(x)| \leqslant P_t^{D,V} |f| (x) \leqslant P_t^{D,-V^-} |f| (x) \leqslant P_t^{-V^-} |f| (x)$$

Hence for all $p \in [1, +\infty]$,

$$\sup_{0 \leqslant t \leqslant 1} \|P_t^{D,V}\|_p \leqslant \sup_{0 \leqslant t \leqslant 1} \|P_t^{-V^-}\|_p < +\infty \tag{2.7}$$

by (2.2) and the assumption that $V^- \in \mathscr{K}$ (see the recalls above).

From the strong continuity of $(P_t^{D,V})$ in $L^p(D, dx)$ $(1 \leqslant p < +\infty)$, we need only to show that $P_t^{D,V} f$ is continuous at $t = 0$ in $L^p(D, dx)$ for f belonging to the dense subset $C_0(D)$ (the space of all continuous functions with compact support) of $L^p(D, dx)$. Fix a such f, since $\mathbf{P}_x(\tau_D > 0) = 1$, $\forall x \in D$,

$$1_{[0 < \tau_D]} f(B_t) \cdot \exp\left(-\int_0^t V(B_s) \, ds \right) \to f(x), \qquad \mathbf{P}_x\text{-a.s. for } x \in D \quad (m\text{-a.e.})$$

as $t \to 0$, by (2.5). By the dominated convergence and our condition on V^-, we get

$$P_t^{D,V} f(x) \to f(x), \qquad m\text{-a.e.}$$

This convergence is then in $L^p(B, m)$ by (2.7) for every Borel subset B of D of finite Lebesgue measure, by the uniform integrability criterion and (2.7). In further, outside of B,

$$1_{B^c} |P_t^{D,V} f(x)| \leqslant 1_{B^c} \|P_t^{-V^-}\|_\infty \|f\|_\infty P_t(x, \operatorname{supp} f)$$

where

$$P_t(x, dy) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right) dy$$

is the transition kernel of the BM. Hence his norm in L^p can be also chosen arbitrarily small for B large enough, we have so established the desired strong continuity.

(b) Let $V^n := \min(V, n)$. It belongs to the Kato's class. In this case, (b) is well known (see [CZ] e.g.). Since $\mathscr{E}_D^{V^n}$ increases to $\mathscr{E}_D^V$, by [RS, Th. 5.14], $P_t^n := P_t^{D, V^n}$ converges strongly to P_t^∞, and (P_t^∞) is a strongly continuous semigroup of bounded self-adjoint operators on $L^2(D, m)$. But for every $0 \leqslant f \in L^2(D, m)$, $P_t^n f$ decreases to $P_t^{D,V} f$. Hence

$$P_t^{D,V} f = P_t^\infty f,$$

proving (b).

(c) Since $V^- \in \mathscr{K}$, it is bounded in form by $-\Delta/2$ with an arbitrary bound, then every core of $\mathscr{E}_D^{V^+}$ is a core of $\mathscr{E}_D^V$. On the other hand, that $C_0^\infty(D)$ is a form core of $\mathscr{E}_D^{V^+}$ follows from [FOT, Theorem 6.1.2] with the underlying Dirichlet form therein given now by

$$\mathscr{E}_D(f, g) = \tfrac{1}{2} \int_D \nabla f \cdot \nabla g \, dx, \qquad \forall f, g \in \mathscr{D}(\mathscr{E}_D) = H_0^{1,2}.$$

The proof is complete. ∎

By the theory of Dirichlet form ([FOT], e.g.), one can prove even that $P_t^{D,V} f(x)$ is quasi-continuous on D. In Lemma 6.3, we shall have a much stronger result in a particular case.

Lemma 2.2 *Let $-A_D$ be the generator of $(P_t^{D,V})_{t \geqslant 0}$ in $L^2(D)$. Then*

$$\begin{aligned} &\mathscr{D}(A_D) = \{ f \in \mathscr{D}(\mathscr{E}_D^V) \mid (-\Delta/2 + V) f \quad (in\ \mathscr{D}'(D)) \in L^2(D)\} \\ &A_D f = (-\Delta/2 + V) f, \qquad \forall f \in \mathscr{D}(A_D) \end{aligned} \tag{2.8}$$

In particular, if $V \in L^2_{loc}(D)$, then $C_0^\infty(D) \subset \mathscr{D}(A_D)$ and A_D is the Frichdrichs extension of the symmetric operator $S = (-\Delta/2 + V, C_0^\infty(D))$ in $L^2(D)$.

Remarks (2.i). Since $V \in L^1_{loc}(D)$ and $f \in \mathscr{D}(\mathscr{E}^V_D) \subset L^2(|V|\,dx)$, by Schwarz inequality, $Vf \in L^1_{loc}(D)$, it is then a distribution. So (2.8) is well defined.

(2.ii) If V^+ belongs also to the Kato class, then $\mathscr{D}(\mathscr{E}^V_D) = \mathscr{D}(\mathscr{E}_D) = H^{1,2}_0$, this lemma is known (see e.g. [CZ, Th. 3.27]).

Proof. Since A_D is the self-adjoint operator determined by the form $\mathscr{E}^V_D$, we have $\mathscr{D}(A_D) \subset \mathscr{D}(\mathscr{E}^V_D)$. As $C_0^\infty(D)$ is a form core for $\mathscr{E}^V_D$, hence for $f \in \mathscr{D}(\mathscr{E}^V_D)$ fixed, it is an element in $\mathscr{D}(A_D)$ iff the linear mapping

$$g \mapsto \mathscr{E}^V_D(f, g), \qquad \forall g \in C_0^\infty(D)$$

is bounded in $L^2(D)$. But by the integration by parts, we have for such f, g,

$$\mathscr{E}^V_D(f, g) = \int_D (-\Delta f/2)\, g\, dx + \int_D Vfg\, dx$$

Therefore $f \in \mathscr{D}(A_D)$ iff $(-\Delta/2 + V) f \in L^2(D)$, giving (2.8a). And (2.8b) follows also from the above formula.

The last claim is obvious, as $C_0^\infty(D)$ is a form core of $\mathscr{E}^V_D$. ∎

Contrary to the situation in L^2, it is much more difficult to describe the domain of $-A^{(1)}_D$, the generator of the strong continuous $(P^{D,V}_t)$ in $L^1(D)$. We are content only of the

Lemma 2.3. *Let $\Delta^{(1)}_D/2$ be the generator of the killed Brownian motion semigroup*

$$P^D_t f(x) := \mathbf{E}^x f(B_t)\, 1_{[t < \tau_D]} \tag{2.9}$$

in $L^1(D)$. If $f \in \mathscr{D}(\Delta^{(1)}_D)$ and $Vf \in L^1(D)$, then $f \in \mathscr{D}(A^{(1)}_D)$, and

$$A^{(A)}_D f = -\Delta^{(1)}_D f/2 + Vf \tag{2.10}$$

In particular, $A^{(1)}_D$ is an (closed) extension of the operator $Sf = (-\Delta/2 + V) f$: $C_0^\infty(D) \to L^1(D)$.

Proof. Let $V^n := (V \wedge n) \vee (-n)$, $n \in \mathbf{N}$, $P^n_t := P^{D, V^n}_t$, $P^\infty_t := P^{D,V}_t$, and A^n the generator of (P^n_t) in $L^1(D)$. It is well known that (by the bounded perturbation)

$$\mathscr{D}(A^n) = \mathscr{D}(\Delta^{(1)}_D) \qquad \text{and} \qquad A^n f = -\Delta^{(1)}_D f/2 + V^n f \qquad \text{for} \quad f \in \mathscr{D}(\Delta^{(1)}_D).$$

For every $f \in \mathscr{D}(\Delta^{(1)}_D) \cap L^1(D, |V|\, dx)$, we have

$$P^n_t f - f = -\int_0^t P^n_s A^n f\, ds, \qquad \forall n \in \mathbf{N}. \tag{2.11}$$

Now letting $n \to +\infty$, we get by the dominated convergence

$$A^n f \to -\Delta_D^{(1)} f/2 + Vf := g \qquad \text{in } L^2(D),$$

and by the Feynman–Kac formula (0.1),

$$P_t^n g \to P_t^\infty g, \qquad \text{in } L^1(D), \quad \forall g \in L^1(D)$$

By (2.7) and the two convergences above,

$$\begin{aligned} h^n(t) := \|P_t^n A^n f - P_t^\infty g\|_1 &\leqslant \|P_t^n A^n f - P_t^n g\|_1 + \|P_t^n g - P_t^\infty g\|_1 \\ &\leqslant \|P_t^n\|_1 \cdot \|A^n f - g\|_1 + \|P_t^n g - P_t^\infty g\|_1 \\ &\leqslant \|P_t^{-V^-}\|_1 \cdot \|A^n f - g\|_1 + \|P_t^n g - P_t^\infty g\|_1 \to 0, \end{aligned}$$

and $\{h^n(t); n \in \mathbf{N}\}$ is uniformly bounded on the compact time intervals. Thus when n tends to infinity, (2.11) becomes

$$P_t^\infty f - f = -\int_0^t P_s^\infty g \, ds.$$

Consequently $f \in \mathscr{D}(A_D^{(1)})$ and $A_D^{(1)} f = g$, the desired result. ∎

2.2. *Several Other Lemmas*

The following lemma is contained in [CFKS, Lemma 2.6],

Lemma 2.4. *If*

$$f \in L^\infty_{loc}(\mathbf{R}^d) \qquad \textit{and} \qquad \Delta f \in L^1_{loc}(\mathbf{R}^d),$$

then $\nabla f \in L^2_{loc}(\mathbf{R}^d)$ *and for every* $g \in C_0^\infty(\mathbf{R}^d)$,

$$\int g \, |\nabla f|^2 \, dx = \tfrac{1}{2} \int f^2 \Delta g \, dx - \int g f \, \Delta f \, dx \tag{2.12}$$

(*the above integrals are taken in* $\mathbf{R}^d$).

Remarks. For $f \in C_0^\infty(\mathbf{R}^d)$, it is trivial that (2.12) holds for $g = 1$, and then

$$\int_{\mathbf{R}^d} |\nabla f|^2 \, dx \leqslant \|f\|_\infty \, \|\Delta f\|_1 \tag{2.13}$$

The following generalized Ito's formula will play a key role, LEMMA 2.5. *If*

$$f \in L^\infty_{loc}(D) \qquad \text{and} \qquad \Delta f \in L^1_{loc}(D),$$

then $\nabla f \in L^2_{loc}(D)$ *and for open* D_n *such that* $\bar{D}_n$ *is a compact subset of* D,

$$M^f_{t \wedge \tau_n} := \tilde{f}(B_{t \wedge \tau_n}) - \tilde{f}(B_0) - \tfrac{1}{2} \int_0^{t \wedge \tau_n} \Delta f(B_s)\, ds = \int_0^{t \wedge \tau_n} \nabla f(B_s)\, dB_s \tag{2.14}$$

is a $\mathbf{P}_\mu$ *square integrable martingale, where* $\tilde{f}$ *is the quasi-continuous version of* f, $\tau_n = \inf\{t \geqslant 0;\, B_t \notin D_n\}$, *and the initial probability measure* μ *is absolutely continuous w.r.t. m such that* $h_\mu := d\mu/dx \in L^\infty(D, m)$.

Remarks (2.iii). Up to $\mathbf{P}_\mu$-indistinguible equivalence, the last two processes in (2.14) do not depend of the Borel versions of Δf, ∇f.

Proof. The first claim is in Lemma 2.4. And we can then choose one quasi-continuous version $\tilde{f}$ of f (i.e. $\tilde{f} = f$ m-a.e. and $\tilde{f}(B_\cdot)$ is $\mathbf{P}_x$-*a.e.* continuous on $[t < \tau_D]$ for m-a.e. $x \in D$, see [Fu]). Turn now to the second claim.

One can construct an open D_{n+1} such that

$$\bar{D}_n \subset D_{n+1} \qquad \text{and} \qquad \bar{D}_{n+1} \text{ is a compact subset of } D,$$

and $u_n \in C_0^\infty(\mathbf{R}^d)$ such that

$$u_n \geqslant 0, \qquad u_n(x) = 1 \qquad \text{for all} \quad x \in D_n \quad \text{and} \quad \operatorname{supp}(u_n) \subset D_{n+1}.$$

For any distribution ψ over D, we can regard ψu_n as a distribution over $\mathbf{R}^d$. We have as distribution over $\mathbf{R}^d$,

$$\Delta(f u_n) = u_n\, \Delta f + f\, \Delta u_n + 2 \nabla f \cdot \nabla u_n, \qquad \nabla(f u_n) = f \nabla u_n + u_n \nabla f$$

Hence $\nabla(f u_n) \in L^2(\mathbf{R}^d)$ by Lemma 2.4, and $\Delta(f u_n)$ belongs to $L^1(\mathbf{R}^d)$. Obviously $\tilde{f} u_n$ is one quasi-continuous version of $f u_n$.

Let $h^\varepsilon(x) = \phi^{-d} h(x/\varepsilon)$ where

$$0 \leqslant h \in C_0^\infty(\mathbf{R}^d), \qquad \operatorname{supp}(h) \subset \{x;\, |x| \leqslant 1\}, \qquad \int_{\mathbf{R}^d} h(x)\, dx = 1.$$

Choose $f^\varepsilon = (f u_n) * h^\varepsilon$ (where $*$ denotes the convolution). We have

$$f^\varepsilon \to f u_n \qquad \text{and} \qquad \nabla f^\varepsilon \to \nabla(f u_n) \quad \text{in } L^2, \qquad \Delta f^\varepsilon \to \Delta(f u_n) \quad \text{in } L^1.$$

By the usual Ito's formula,

$$M_t^\varepsilon := f^\varepsilon(B_t) - f^\varepsilon(B_0) - \tfrac{1}{2}\int_0^t \Delta f^\varepsilon(B_s)\,ds = \int_0^t \nabla f^\varepsilon(B_s)\,dB_s \tag{2.15}$$

is a $\mathbf{P}_\mu$-square integrable continuous martingale. And

$$\mathbf{E}^\mu \left\langle M^\varepsilon - \int_0^\cdot \nabla(fu_n)(B_s)\,dB_s \right\rangle_t = \mathbf{E}^\mu \int_0^t |\nabla(f^\varepsilon - fu_n)|^2\,(B_s)\,ds$$

$$\leqslant \|h_\mu\|_\infty \int_{\mathbf{R}^d} |\nabla(f^\varepsilon - fu_n)|^2\,dx \to 0.$$

For every $t \geqslant 0$ fixed, the left side of (2.15) converges in $L^1(\mathbf{P}_\mu)$ to

$$M_t^{fu_n} := \tilde{f}u_n(B_t) - \tilde{f}u_n(B_0) - \tfrac{1}{2}\int_0^t \Delta(fu_n)(B_s)\,ds$$

Hence

$$M_t^{fu_n} = \int_0^t \nabla(fu_n)(B_s)\,dB_s$$

$\mathbf{P}_\mu$-*a.s.*. But these two processes are continuous, they are then $\mathbf{P}_\mu$-indistinguable.

As $M^{fu_n}_{\tau_n \wedge \cdot} = M^f_{\tau_n \wedge \cdot}$, we get (2.14), the desired result. ▮

Remarks (2.iv). This lemma is a particular case of the Fukushima's martingale decomposition for $f \in H^{1,2}_{loc}$, which is satisfied by Lemma 2.4. And it still holds for $f \in L^p_{loc}(D)$, $\Delta f \in L^q_{loc}(D)$ for $1/p + 1/q = 1$.

Finally we shall require a functional analysis lemma which might be known.

LEMMA 2.6. *Let $(T_t)_{t\geqslant 0}$ be a C_0-semigroup on the Banach space $(X, \|\cdot\|)$. Let $-A$ be its generator with its domain $\mathscr{D}(A)$, and $\mathscr{D}$ a fixed subset of $\mathscr{D}(A)$, dense in X. Let*

$$\lambda_0 := \lim_{t \to +\infty} \frac{1}{t} \log\ \|T_t\|. \tag{2.16}$$

We have equivalence between

(i) *the closure of $S := A|_{\mathscr{D}}$ is A (we say that $-S$ or S is an essential generator on X);*

(ii) *for all $\lambda > \lambda_0$, $(\lambda + A)(\mathscr{D})$ is dense in X;*

(iii) *for some $\lambda \in \rho(A)$ (the resolvant set of A), $(\lambda + A)(\mathscr{D})$ is dense in X.*

In this situation (T_t) *is the unique* C_0*-semigroup on* X *such that its generator is an extension of* $A|_{\mathscr{D}}$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, as the resolvant $R_\lambda := (\lambda + A)^{-1}$ is defined on X and bounded for $\lambda > \lambda_0$. To prove (iii) $\Rightarrow$ (i), let $f \in \mathscr{D}(A)$. By (iii), we can find $f_n \in \mathscr{D}$ such that $(\lambda + A) f_n \to (\lambda + A) f$ in X. By the boundedness of R_λ, $f_n \to f$ in X, too. These in turn imply $Af_n \to Af$, the desired (i).

For the last claim, let (T'_t) be an arbitrary such semigroup with its generator A'. As $(\lambda + A')(\mathscr{D}) = (\lambda + A)(\mathscr{D})$ is dense in X $(\lambda > \lambda_0)$, by the equivalence above, A' is the closure of $A|_{\mathscr{D}}$. Hence $A' = A$, as desired.

Remarks (2.v). The author do not know to prove that the last uniqueness statement is equivalent to (i). This might be valid.

3. PROOF OF THEOREM 1.1.

3.1. *Proof of Theorem* 1.1 *Except* (iv)

The first claim follows from Lemma 2.1(a) and Lemma 2.3. The last claim about the uniqueness of semigroup follows from Lemma 2.6. We turn now to the equivalence of (i), (ii), (iii). And the equivalence of (iii) and (iv) is left to 3.2.

(i) $\Leftrightarrow$ (ii): By Lemma 2.6, they are all equivalent to the fact that $(\lambda + S)(C_0^\infty(D))$ is dense in $L^1(D)$ for some λ large enough.

We turn to the main (iii) $\Rightarrow$ (i):

Step 1. Write

$$e_t(V) = \exp\left(-\int_0^t V(B_s)\, ds\right), \qquad \forall t \geqslant 0.$$

By the assumption that $V^- \in \mathscr{K}$ and (2.2), we can find some $\alpha > 0$ and $C > 0$ such that for all $t > 0$,

$$\sup_{x \in \mathbf{R}^d} \mathbf{E}_x e_t(-2V^-) < Ce^{(2\alpha - 1)\, t}. \tag{3.1a}$$

Thus for any finite stopping time τ,

$$\begin{aligned} \sup_{x \in \mathbf{R}^d} \mathbf{E}_x e_\tau(2\alpha - 2V^-) &\leqslant \sup_{x \in \mathbf{R}^d} \mathbf{E}_x \sum_{k=0}^{+\infty} 1_{[k \leqslant \tau < k+1]} e_{k+1}(-2V^-)\, e^{-2\alpha k} \\ &\leqslant \sum_{k=0}^{+\infty} Ce^{(2\alpha - 1)(k+1)} e^{-2\alpha k} < +\infty. \end{aligned} \tag{3.1b}$$

Step 2. By Lemma 2.6, it is enough to prove that $(\lambda+S)(C_0^\infty(D))$ is dense in $L^1(D)$, where $\lambda=\alpha+1$. As the dual of L^1 is L^∞, this is equivalent to

$$\forall f \in L^\infty(D): \qquad (\lambda-\Delta/2+V)\, f=0 \qquad (\text{in } \mathscr{D}'(D)) \Rightarrow f=0 \tag{3.2}$$

Fix a such function f. Let (D_n) be an increasing sequence of open subsets of D such that $\bar{D}_n$ is a compact subset of D_{n+1} $(n \in \mathbf{N})$, and $\tau_n := \inf\{t \geqslant 0; B_t \notin D_n\}$, as in Lemma 2.5.

Note that $\Delta f/2=(\lambda+V)\, f \in L^1_{loc}(D)$. By the key Lemma 2.5, $M^f_{\tau_n \wedge \cdot}$ defined in (2.14) is a $\mathbf{P}_\mu$ square integrable martingale, where $\mu=h_\mu m$ with $0<h_\mu \in L^\infty$. We get by Ito's formula that $\mathbf{P}_\mu$-*a.s.*, for all $t \leqslant \tau_n$

$$\begin{aligned}\tilde{f}(B_t)\, e_t(\lambda+V) &= \tilde{f}(B_0) - \int_0^t \tilde{f}(B_s)(\lambda+V)(B_s)\, e_s(\lambda+V)\, ds \\ &\quad + \int_0^t e_s(\lambda+V)(dM^f_s + \Delta f(B_s)/2\, ds) \\ &= \tilde{f}(B_0) + \int_0^t e_s(\lambda+V)\, dM^f_s\end{aligned}$$

which, stopped at τ_n, is a $\mathbf{P}_\mu$-local martingale.

Step 3. Let $C=\|f\|_\infty$. Then $|\tilde{f}(x)| \leqslant C$ quasi everywhere on D (see [FOT, Lemma 4.1.5]), we get hence by Schwartz inequality,

$$\begin{aligned}\mathbf{E}^\mu \sup_{t \geqslant 0} [\tilde{f}(B_{t \wedge \tau_n})\, e_{t \wedge \tau_n}(\lambda+V)]^2 &\leqslant C^2 \mathbf{E}^\mu e_{\tau_n}(2+2V^+) \cdot \mathbf{E}^\mu e_{\tau_n}(2\alpha-2V^-) \\ &< +\infty\end{aligned} \tag{3.3}$$

by (3.1b).

Step 4. By Step 3, the local martingale stopped at τ_n obtained in Step 2 is actually a martingale. Hence

$$\tilde{f}(B_0)=\mathbf{E}^{B_0} e_{\tau_n}(\lambda+V)\, \tilde{f}(B_{\tau_n}). \tag{3.4}$$

Letting $n \to +\infty$, obviously $\tau_n \to \tau_D$, and by (C1),

$$\mathbf{E}^\mu e_{\tau_n}(2+2V^+) \to 0. \tag{3.5}$$

Hence we get by (3.3),

$$\mathbf{E}^\mu(\tilde{f}(B_0))^2=0,$$

which is the desired result (3.2).

(i) $\Rightarrow$ (iii): Take $\lambda = 1 + \alpha$ where α satisfies (3.1) and $\alpha > \lambda_0$, where λ_0 is defined in Lemma 2.6. with $T_t = P_t^{D,V}$, $X = L^1(D)$. By absurd and Lemma 2.6, we should prove that if (C1) is not satisfied, then there is one $0 \neq f \in L^\infty(D)$ such that $(\lambda - \Delta/2 + V) f = 0$ in $\mathscr{D}'(D)$.

Inspired by (3.4), we choose

$$f(x) := \mathbf{E}^x 1_{[\tau_D < +\infty]} e_{\tau_D}(\lambda + V) \tag{3.6a}$$

$$f_n(x) := \mathbf{E}^x 1_{[\tau_D < +\infty]} e_{\tau_D}(\lambda + V^n) \tag{3.6b}$$

where $V^n = V \wedge n$. Since (C1) is not satisfied (assumption), $m(f > 0) > 0$. By (3.1) and (3.3), $f_n \in L^\infty(D)$ and $f_n(x)$ decreases to $f(x)$ by the dominated convergence.

By the strong Markov property, for any ball $R \subset D$ containing x,

$$f_n(x) = \mathbf{E}^x e_{\tau_R}(\lambda + V^n)\, f_n(B_{\tau_R}) \tag{3.7}$$

As $V^n \in \mathscr{K}$, we can apply [CZ, Corollary of Th. 5.18, p. 150] to get: f_n *is continuous and bounded such that*

$$(\lambda - \Delta/2 + V^n)\, f_n = 0 \qquad (\text{in } \mathscr{D}'(R))$$

Therefore, for any $g \in C_0^\infty(R)$, we have by dominated convergence,

$$\langle f, (\lambda - \Delta/2 + V)\, g \rangle = \lim_{n \to \infty} \langle f_n, (\lambda - \Delta/2 + V^n)\, g \rangle = 0.$$

In other words $(\lambda - \Delta/2 + V) f = 0$ in $\mathscr{D}'(R)$. As R is arbitrary, this holds also in $\mathscr{D}'(D)$. ∎

Remarks (3.i). The function $f(x)$ defined by (3.6), called the gauge function according to Chung-Zhao [CZ], plays a very important role in the studies of Schrödinger operators. But almost all important properties of the gauge function in [CZ] depends heavily on the assumption that $V \in \mathscr{K}$ (globally), under which (C1) is trivially violated once $\tau_D < +\infty$ has a positive probability. Notice also that the key relation (3.4) (even valid for every initial state) is established for $V \in \mathscr{K}_{loc}$ in [CZ, Th. 5.21 and Th. 4.15] by a completely different approach (which seems not very well adapted to the general situation here).

3.2.

For Theorem 1.1, it remains to show (C1) $\Leftrightarrow$ (iv) under the auxiliary condition that $V^+ \in \mathscr{K}_{loc}(D)$ and D is connected. One consequence of it is

$$\mathbf{P}_x\left(\int_0^t V^+(B_s)\, ds < +\infty, \forall t < \tau_D\right) = 1, \qquad \forall x \in D \tag{3.8}$$

(instead of $\langle\langle dx$-a.e. $x \in D\rangle\rangle$ under (H1) only). As only V^+ is concerned, we shall assume $V = V^+ \geqslant 0$ in this paragraph.

We begin by a remark. If D is connected and $V \in \mathscr{K}_{loc}(D)$, (C1) is equivalent to

$$\exists x_0 \in D: \qquad \mathbf{P}_{x_0}\left(\int_0^{\tau_D} V(B_s)\, ds = +\infty\right) = 1 \tag{C1$'$}$$

and to

$$\forall x_0 \in D: \qquad \mathbf{P}_{x_0}\left(\int_0^{\tau_D} V(B_s)\, ds = +\infty\right) = 1 \tag{C1$''$}$$

In fact, (C1$''$) $\Rightarrow$ (C1) $\Rightarrow$ (C1$'$) are obvious. To see (C1$'$) $\Rightarrow$ (C1$''$), take a small open ball B centered at x_0 such that $\bar{B} \subset D$. Let σ be the normalized surface area measure on the sphere ∂B. As

$$\int_0^{\tau_B} V^+(B_s)\, ds < +\infty, \qquad \mathbf{P}_{x_0}\text{-a.s.}$$

by (3.8), we have by the strong Markov property and the fact that $\mathbf{P}_{x_0}(B_{\tau_B} \in dz) = \sigma(dz)$,

$$\begin{aligned} h(x_0) := \mathbf{P}_{x_0}&\left(\int_0^{\tau_D} V^+(B_s)\, ds = +\infty\right) \\ &= \int_{\partial B} \mathbf{P}_z\left(\int_0^{\tau_D} V^+(B_s)\, ds = +\infty\right) \sigma(dz) \\ &= \mathbf{P}_\sigma\left(\int_0^{\tau_D} V^+(B_s)\, ds = +\infty\right) \end{aligned} \tag{3.9}$$

Now for every $x \in B$, as $\mathbf{P}_x(B_{\tau_B} \in dz)$ is absolutely continuous w.r.t. $\sigma(dz)$, we get by (C1$'$) and (3.9) that $\int_0^{\tau_D} V^+(B_s)\, ds = +\infty$, $\mathbf{P}_x$-*a.s.* By extension and the connectness of D, (C1$''$) holds.

We can now proceed the proof of

(iii) or (C1) $\Rightarrow$ ((iv) with $\forall$): If in contrary $\exists x_0 \in D$, there is a Borel subset O charged by the BM until τ_D such that $\int_O G_D(x_0, x)(1+V)(x)\, dx < +\infty$, i.e.,

$$\mathbf{E}^{x_0} \int_0^{\tau_D} 1_O(V+1)(B_t)\, dt < +\infty$$

Let $\tau < \tau_D$ be the random time associated to O and $\mathbf{P}_{x_0}$. We have $\mathbf{P}_{x_0}$-*a.s.*

$$\int_0^{\tau_D} (1+V)(B_s)\, ds = \int_0^{\tau} (1+V)(B_s)\, ds + \int_{\tau}^{\tau_D} (1+V)(B_s)\, ds < +\infty$$

over $[B_t \in O, \forall \tau < t < \tau_D]$. As this last set has a positive $\mathbf{P}_{x_0}$-probability, $(\mathrm{C1}'')(\Leftrightarrow (\mathrm{C1}))$ is violated.

(iv) $\Rightarrow$ (C1): If (C1) is not satisfied, consider the Gauge function $f(x)$ defined in (3.6a) with $\lambda = 1$. As in the implication (i) $\Rightarrow$ (iii), f satisfies (3.7) and then it is continuous on D, by [CZ].

By the strong Markov property, for any stopping time T

$$Z_T := f(B_{\tau \wedge \tau_D})\, e_{T \wedge \tau_D}(1+V) = \mathbf{E}(1_{[\tau_D < \infty]} e_{\tau_D}(1+V) \mid \mathscr{F}_{T \wedge \tau_D}) \tag{3.10}$$

then Z is a martingale w.r.t. $\mathbf{P}_x$, $\forall x \in D$. Since Z is bounded by one, by martingale convergence, when t increases to τ_D, we have $\mathbf{P}_x$-a.s.,

$$Z_t \to 1_{[\tau_D < \infty]} e_{\tau_D}(1+V) = e_{\tau_D}(1+V),$$

which gives us by (3.10) and the dominated convergence

$$f(B_t) \to 1, \qquad \mathbf{P}_x\text{-a.s. over } A = \left[\int_0^{\tau_D} (1+V)(B_s)\, ds < +\infty \right] \tag{3.11}$$

Let $O = [f > 1/2]$, an open set in D, and define

$$\tau = \mathrm{supp}\{t \geqslant 0;\, B_t \in D \backslash O\} \text{ on } A, \qquad \text{and} \qquad = 0 \text{ on } A^c$$

We have $\tau < \tau_D$ by (3.11) and the continuity of f. As $\mathbf{P}_x(A) > 0$ by our absurd assumption and $(\mathrm{C1}) \Leftrightarrow (\mathrm{C1}')$, we get by (3.11),

$$\mathbf{P}_x(B_t \in O;\, \forall t \in (\tau, \tau_D)) > 0,$$

i.e., O is charged by the BM until τ_D.

It remains to calculate

$$\begin{aligned}
&\int_O G_D(x_0, x)(1+V(x))\, dx \\
&\qquad \leqslant 2\mathbf{E}^{x_0} \int_0^{\tau_D} f(B_t)(1+V)(B_t)\, dt \\
&\qquad = 2\mathbf{E}^{x_0} \int_0^{\tau_D} (1+V)(B_t) \exp\left(- \int_t^{\tau_D} (1+V)(B_s)\, ds \right) dt \leqslant 2
\end{aligned}$$

where the last inequality follows from the Newton-Leibnitz formula. This is in contradiction with (1.2) in (iv). The proof of Theorem 1.1 is completed. ▮

4. PROOF OF THEOREM 1.4.

4.1. *Proof of* (*a*)

By Lemma 2.1(c), A_D is the smallest element of $\mathscr{A}_+(S)$, then of $\mathscr{A}_M(S)$ too. It remains to prove that A_R, the s.a. operator associated to $\mathscr{E}_R^V$ is the largest element of $\mathscr{A}_M(S)$. It is well known that it is a Dirichlet form, i.e., $A_R^V \in \mathscr{A}_M(S)$. Hence we should show that for any $A \in \mathscr{A}_M(S)$,

$$\forall f \in \mathscr{D}(\mathscr{E}_A), \qquad \text{then} \quad f \in \mathscr{D}(\mathscr{E}_R^V) \qquad \text{and} \qquad \mathscr{E}_A(f, f) \geqslant \mathscr{E}_R^V(f, f), \tag{4.1}$$

Note at first a general fact: the dual of S in $L^2(D)$ is given by

$$\begin{aligned} \mathscr{D}(S^*) &= \{ f \in L^2(D); (-\Delta f/2 + Vf) \in L^2(D) \} \\ S^* f &= (-\Delta f/2 + Vf), \qquad \forall f \in \mathscr{D}(S^*) \end{aligned} \tag{4.2}$$

We prove now (4.1) by four steps.

Step 1. $\mathscr{D}(A) \cap L^\infty \cap L^1$ is an operator core (then a form core) of A.

In fact, $\mathscr{D}(A) = R_\alpha^A(L^2)$ for any $\alpha > 0$ fixed, where $R_\alpha^A := (\alpha + A)^{-1}$ is the resolvant. As $L^\infty \cap L^1$ is a dense subset of L^2, then $R_\alpha^A(L^\infty \cap L^1)$ is an operator core of A. But by the sub-Markov property of αR_α^A, $R_\alpha^A(L^\infty \cap L^1)$ is contained in $\mathscr{D}(A) \cap L^\infty \cap L^1$.

Step 2. By Step 1, it is enough to prove (4.1) for those $f \in \mathscr{D}(A) \cap L^\infty \cap L^1$. The following key equalities, borrowed from [FOT, (3.3.27)], are relied only on the sub-Markov property,

$$\mathscr{E}_A(f, f) = \lim_{\beta \to +\infty} \uparrow \mathscr{E}_A^{(\beta)}(f, f) \tag{4.3a}$$

where

$$\mathscr{E}_A^{(\beta)}(f, f) := \beta \langle f - \beta R_\beta^A f, f \rangle = \tfrac{1}{2} \langle f_\beta, 1 \rangle \tag{4.3b}$$

$$\begin{aligned} f_\beta(x) &:= \beta^2 R_\beta^A (f(x) - f)^2 (x) + 2\beta f(x)^2 (1 - \beta R_\beta^A 1(x)) \\ &= -\beta(f^2 - \beta R_\beta^A f^2) + 2\beta f(f - \beta R_\beta^A f) + \beta f^2 (1 - \beta R_\beta^A 1) \end{aligned} \tag{4.3c}$$

For every $\phi \in C_0^\infty(D)$ such that $0 \leqslant \phi \leqslant 1$, since $f_\beta \geqslant 0$ by the first line in (4.3c), we have

$$\begin{aligned}\mathscr{E}_A^{(\beta)}(f, f) &\geqslant \tfrac{1}{2}\langle f_\beta, \phi\rangle \\ &= \tfrac{1}{2}\langle -\beta(f^2 - \beta R_\beta^A f^2), \phi\rangle + \langle \beta f(f - \beta R_\beta^A f), \phi\rangle \\ &\quad + \tfrac{1}{2}\langle f^2\phi, \beta(1 - \beta R_\beta^A 1)\rangle \end{aligned} \tag{4.4}$$

Step 3. Now we turn to study the three terms in (4.4), when β goes to infinity. As A is an extension of S and $f \in L^\infty$, we have $\phi \in \mathscr{D}(A)$ and

$$\begin{aligned}\langle \beta(f^2 - \beta R_\beta^A f^2), \phi\rangle &= \langle f^2, \beta(\phi - \beta R_\beta^A \phi)\rangle \to \langle f^2, A\phi\rangle \\ &= \langle f^2, S\phi\rangle = \langle f^2, -\Delta\phi/2\rangle + \langle f^2, V\phi\rangle. \end{aligned} \tag{4.5}$$

For the second term in (4.4), as $f \in \mathscr{D}(A) \cap L^\infty$,

$$\beta(f - \beta R_\beta^A f) \to Af \qquad \text{in } L^2(D).$$

As $A \subset S^*$, it follows that

$$\begin{aligned}\langle \beta f(f - \beta R_\beta^A f), \phi\rangle &\to \langle Af, f\phi\rangle = \langle S^* f, f\phi\rangle \\ &= \langle -\Delta f/2, f\phi\rangle + \langle f^2, V\phi\rangle. \end{aligned} \tag{4.6}$$

For the last and the third term in (4.4), notice at first that $f \in \mathscr{D}(A) \cap L^\infty \cap L^1 \subset \mathscr{D}(S^*)$ implies $\Delta f \in L^1_{loc}(D)$ and $\Delta(f^2\phi) \in L^1(D)$. By Lemma 2.4, $\nabla(f^2\phi) \in L^2(D)$. Let $h^\varepsilon = \varepsilon^{-d} h(x/\varepsilon)$, where

$$0 \leqslant h \in C_0^\infty(\mathbf{R}^d), \qquad \operatorname{supp}(h) \subset \{x;\, |x| \leqslant 1\}, \qquad \int_{\mathbf{R}^d} h(x)\, dx = 1.$$

Take

$$g^\varepsilon = (f^2\phi) * h^\varepsilon$$

For $\varepsilon > 0$ sufficiently small, $g^\varepsilon \in C_0^\infty(D)$ and

$$g^\varepsilon \to f^2\phi, \qquad \Delta g^\varepsilon \to \Delta(f^2\phi), \qquad Vg^\varepsilon \to Vf^2\phi$$

In $L^1(D)$. In other words, $f^2\phi$ belongs to the domain of the closure $\bar{S}^{(1)}$ of S in $L^1(D)$ (recall S is closable in $L^1(D)$ by Lemma 2.3), and

$$\bar{S}^{(1)}(f^2\phi) = -\tfrac{1}{2}\Delta(f^2\phi) + V(f^2\phi).$$

Let $A^{(1)}$ be the generator of (T_t^A) in $L^1(D)$. As $C_0^\infty(D) \subset \mathscr{D}(A)$, and

$$A(C_0^\infty(D)) = S(C_0^\infty(D)) \subset L^2 \cap L^1$$

$C_0^\infty(D) \subset \mathscr{D}(A^{(1)})$. Therefore $\bar{S}^{(1)} \subset A^{(1)}$. As $f^2\phi \in \mathscr{D}(\bar{S}^{(1)}) \subset \mathscr{D}(A^{(1)})$, we get hence

$$\beta(f^2\phi - \beta R_\beta^A(f^2\phi)) \to A^{(1)}(f^2\phi) = \bar{S}^{(1)}(f^2\phi), \qquad \text{in } L^1(D)$$

This leads to

$$\begin{aligned}\langle f^2\phi, \beta(1-\beta A_\beta^A 1)\rangle &= \langle \beta(f^2\phi - \beta R_\beta^A(f^2\phi)), 1\rangle \to \langle \bar{S}^{(1)}(f^2\phi), 1\rangle \\ &= \int_D -\Delta(f^2\phi)/2dx + \int_D Vf^2\phi\, dx \\ &= \int_D Vf^2\phi\, dx \qquad (4.7)\end{aligned}$$

as β tends to infinity.

Step 4. By Lemma 2.4,

$$\tfrac{1}{2}\langle f^2, \Delta\phi\rangle - \langle \Delta f, f\phi\rangle = \int_D |\nabla f|^2\, \phi\, dx$$

Substituting (4.5), (4.6) and (4.7) into (4.4) and using the above formula, we get

$$\lim_{\beta \to +\infty} \mathscr{E}_A^{(\beta)}(f, f) \geqslant \tfrac{1}{2}\int_D |\nabla f|^2\, \phi\, dx + \int_D f^2 V\phi\, dx$$

Taking $0 \leqslant \phi_n \leqslant 1$ in $C_0^\infty(D)$ increasing to 1, we get (4.1) for $f \in \mathscr{D}(A) \cap L^\infty \cap L^1$. We have so finished the proof of (a). ∎

4.2. *Proof of Theorem* 1.4(b)

Before its proof, recall the following facts for D Lipchizian:

Let $\mathscr{E}_R = \mathscr{E}_R^{V=0}$, which is a regular Dirichlet form satisfying local property on $\bar{D}$ ([FOT, Example 1.6.1], [Ma]). Then it corresponds to a continuous Hunt diffusion, called the reflecting Brownian motion (RBM). Bass and Hsu [BH1, 2] (for D bounded) and Fukushima and Tomisaki [FT] (for D unbounded) prove in further that the corresponding RBM Brownian motion (B_t^R) on $\bar{D}$ *satisfying*

(i) its transition kernel semigroup (P_t^R) is Markov, Strong Feller (i.e., $P_t^R f \in C_b(\bar{D})$ for every bounded measurable function f on $\bar{D}$), and $P_t^R(x, dy)$ is absolutely continuous w.r.t. the Lebesgue measure.

(ii) It satisfies the Skorohod representation:

$$B_t^R = B_t + \int_0^t \mathbf{n}(B_t^R)\, dL_t^{\partial D}$$

where $L_t^{\partial D}$ is the local time at ∂D, $\mathbf{n}(\cdot)$ is the inner normal vector at ∂D.

By the regularity of $\mathscr{E}_R$, we can apply the powerful theory of Dirichlet forms in [Fu], [FOT], [MR] etc.

LEMMA 4.1. (a) *$C_0^\infty(D)$ is $\mathscr{E}_R$-dense in $\mathscr{D}_R^0 := \{f \in \mathscr{D}(\mathscr{E}_R) = H^{1,2}(D);$ $\tilde{f} = 0$, $\mathscr{E}_R$-q.e. on $\partial D\}$, where $\tilde{f}$ is the $\mathscr{E}_R$-quasi-continuous version of f* ([FOT, *Lemma* 2.3.4(ii)]). *In particular*, $H_0^{1,2}(D) = \mathscr{D}_R^0$.

(b) *If f is B^R-finely continuous, then f is $\mathscr{E}_R$-quasi-continuous* (*see* [FOT, *Th.* 4.6.1]).

(c) *$O \subset \bar{D}$ is quasi-open iff O equals to a finely open set up to the quasi everywhere equivalence* (*see* [FOT, *Th.* 4.6.1]).

With exactly the same proof as Lemma 2.1(b), we can prove

LEMMA 4.2. *Assume* (H2) *and D is Lipchizian. The sub-Markovian symmetric semigroup corresponding to Dirichlet form $\mathscr{E}_R^V$ is given by the reflecting Feynman–Kac semigroup*

$$P_t^{R,V} f(x) := \mathbf{E}^x \exp\left(-\int_0^t V(B_s^R)\, ds\right) f(B_t^R) \tag{4.8}$$

We turn now to the job.

(b.i) $\Rightarrow$ (b.ii): Since $C_0^\infty(D)$ is a core of $\mathscr{E}_D^V$ (by Lemma 2.1), which equals to $\mathscr{E}_R^V$ by (b.i), hence $C(\bar{D}) \cap \mathscr{D}(\mathscr{E}_R^V)$ is a form core of $\mathscr{E}_R^V$. Prove now (1.10) by absurd. Assume in contrary that there are $z \in \partial D$ and $r > 0$ such that

$$\int_{B(z,r)} V(x)\, dx < +\infty.$$

Take $f \in C_b^2(\bar{D})$ such that $f(x) > 0$, $\forall x \in \partial D \cap B(z, r)$ and $f = 0$ on $D \backslash B(z, r)$. Then $f \in H^{1,2}(D) \cap L^2(V\,dx) = \mathscr{D}(\mathscr{E}_R^V)$. But $\partial D \cap B(z, r)$ has a strictly positive capacity-(1,2) w.r.t. B^R. Then $f \notin H_0^{1,2}(D)$, by Lemma 4.1(a). This contradicts with (b.i).

(b.ii) $\Rightarrow$ (b.i): Since $C(\bar{D}) \cap \mathscr{D}(\mathscr{E}_R^V)$ is assumed to be a core of $\mathscr{E}_R^V$, it is enough to show that

$$C(\bar{D}) \cap \mathscr{D}(\mathscr{E}_R^V) \subset H_0^{1,2}(D).$$

This is equivalent to: if $f \in C(\bar{D}) \cap H^{1,2}(D)$ *such that* $f \notin H_0^{1,2}(D)$, then

$$\int_D f^2 V\,dx = +\infty.$$

For such f, by Lemma 4.1.a, there is $z_0 \in \partial D$ such that $f(z_0) \neq 0$. Now the condition (1.10) in (b.ii) implies the infinity of the integral above.

(b.i) $\Rightarrow$ (b.iii): We begin by some preparations by following Sturm [Stu] and Getoor [Ge]. Introduce

$$\tau_V := \inf\left\{ t > 0;\ \int_0^t V(B_s^R)\,ds = +\infty \right\} \tag{4.9}$$

$$D_V := \{ x \in \bar{D}\colon \mathbf{P}^x(\tau_V > 0) = 1 \} \tag{4.10}$$

$$A_t := \lim_{\varepsilon \downarrow 0} \int_0^{t+\varepsilon} V(B_s^R)\,ds, \qquad M_t := e^{-A_t} \tag{4.11}$$

$$\tilde{P}_t^{R,V} f(x) := \mathbf{E}^x M_t f(B_t^R) \tag{4.12}$$

By Getoor [Ge, Prop. 4.3], (M_t) is an exact multiplicative functional. By Sharpe [Sh, (56.9) in Prop. (56.5)], for every Borel bounded f on $\bar{D}$,

$$G_\alpha f(x) := \int_0^{+\infty} e^{-\alpha t} \tilde{P}_t^{R,V} f(x)\,dt \tag{4.13}$$

is B^R-finely continuous on $\bar{D}$, $\forall \alpha > 0$. Note that $D_V := [G_\alpha 1 > 0]$ is B^R-finely open.

Note for every $t > 0$ fixed,

$$P_t^{R,V} f = \tilde{P}_t^{R,V} f, \qquad dx\text{-a.e.} \tag{4.14}$$

Take now a strictly positive $f \in C_b(\bar{D}) \cap L^1(D)$. Since $G_\alpha f \in \mathscr{D}(\mathscr{E}_R^V)$, being finely continuous, is $\mathscr{E}_R$-*quasi-continuous*, then by (b.i)

$$Cap^R([G_\alpha f > 0] \cap \partial D) = Cap^R(D_V \cap \partial D) = 0, \tag{4.15}$$

where Cap^R is the (1,2)-capacity associated to $\mathscr{E}_R$. Hence $\sigma(D_V \cap \partial D) = 0$, which is justly (b.iii).

(b.iii) $\Rightarrow$ (b.iv): This is obvious as $\mathbf{P}_{x_0}(B_{\tau_D}^R \in dz) = \mathbf{P}_{x_0}(B_{\tau_D} \in dz)$ is absolutely continuous w.r.t. the surface area measure σ of ∂D (see [CZ]).

(b.iv) $\Rightarrow$ (b.i): By (b.iv), we have $\tau_V = \tau_D$, $\mathbf{P}_x$-*a.s.* for dx-a.e. $x \in D$. By (4.14), $\forall t > 0$ fixed, we have dx-a.e. $x \in D$,

$$\begin{aligned}
P_t^{R,V}f(x) &= \mathbf{E}^x f(B_t^R)\, M_t \\
&= \mathbf{E}^x 1_{[t<\tau_V]} f(B_t^R) \exp\left(-\int_0^t V(B_s^R)\, ds\right) \\
&= \mathbf{E}^x 1_{[t<\tau_D]} f(B_t^R) \exp\left(-\int_0^t V(B_s^R)\, ds\right) \\
&= \mathbf{E}^x 1_{[t<\tau_D]} f(B_t) \exp\left(-\int_0^t V(B_s)\, ds\right) \\
&= P_t^{D,V}f(x)
\end{aligned}$$

Hence (b.i) holds.

We have so completed the cycle between (b.i) → (b.iv).

(b.iii) ⇒ (b.v): If in contrary there were a quasi-open subset O of $\bar{D}$ such that $\sigma(O \cap \partial D) > 0$ and $\int_O V(x)\, dx < +\infty$. Then at first for dx-a.e. and consequently for Cap^R-quasi-everywhere $x \in O$,

$$\mathbf{P}_x\left(\int_0^\varepsilon V(B_t^R) \cdot 1_{[B_t^R \in O]}\, dt < +\infty, \forall \varepsilon > 0\right) = 1,$$

i.e., $Cap^R(O \backslash D_V) = 0$ (see [St]). Then $\sigma(\partial D \cap (O \backslash D_V)) = 0$. This is in contradiction with (b.iii).

(b.v) ⇒ (b.iii): If in contrary (b.iii) is false, then for any strictly positive $f \in C_b^2(\bar{D}) \cap L^1(D)$,

$$\sigma([G_\alpha f > 0] \cap \partial D) = \sigma(D_V \cap \partial D) > 0.$$

Choose next $\delta > 0$ such that

$$\sigma([G_\alpha f > \delta] \cap \partial D) > 0.$$

On the other hand, by *Newton–Leibniz formula*,

$$\begin{aligned}
0 &= \mathbf{E}^x \exp\left(-\int_0^\infty (\alpha + V)(B_s^R)\, ds\right) \\
&= 1 - \int_0^\infty \mathbf{E}^x(\alpha + V)(B_t^R) \exp\left(-\int_0^t (\alpha + V)(B_s^R)\, ds\right) dt \\
&= 1 - G_\alpha(\alpha + V)(x), \qquad dx\text{-a.e.}
\end{aligned}$$

we have

$$\langle G_\alpha f, (\alpha + V)\rangle = \langle f, G_\alpha(\alpha + V)\rangle = \langle f, 1\rangle$$

which is finite. Hence for $O=[G_\alpha f>\delta]$,

$$\delta\int_O V\,dx \leqslant \langle G_\alpha f, (\alpha+V)\rangle < +\infty$$

a contradiction with (b.v). We have completed the proof of Theorem 1.4. ∎

Remarks (4.i). The proof of the key part (a) of this theorem follows the ideas in the previous remarkable works [AK], [Ta], [FOT], [RZ1, 2] etc. (which treat the singular diffusions rather than the Schrödinger operators here), but technically more elementary. Especially we have avoided the use of Klein maximal extension (in $\mathscr{A}(S)$), and of the hypoellipcity which plays an important technical role in [Ta], [FOT]. Note also that the main new point here is the studies of the third term in (4.4).

5. PROOF OF PROPOSITION 1.2 AND 1.5

In this section we shall use simple stochastic calculus to verify the conditions (C1) and (C2).

5.1.

All are based on the following

LEMMA 5.1. *Let $V\colon \mathbf{R}\to[0,+\infty]$ be a Borel function such that $V(x)=0$, $\forall x<0$ and $V\in L^1_{loc}(\mathbf{R}\backslash\{0\})$. Let (X_t) be a continuous semimartingale defined on $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbf{P})$ with the Doob-Meyer decomposition $X=X_0+M+A$. Assume there are constant $c, C>0$ such that*

(i) $A_t=\int_0^t a_s\,ds$ *with* $|a_s|\leqslant c$, *a.s.*;
(ii) $\langle M\rangle_t=\int_0^t \sigma_s^2\,ds$ *with* $1/C\leqslant|\sigma_s|\leqslant C$, *a.s.*

We have

(a) *if* $\mathbf{P}(X_0=0)=1$, *then*

$$\mathbf{P}\left(\int_0^\varepsilon V(X_s)\,ds=+\infty, \forall\varepsilon>0\right)=0 \text{ or } 1 \tag{5.1}$$

And the quantity of (5.1) *equals to* 1 *iff*

$$\int_0^1 V(r)\,dr=+\infty \tag{5.2}$$

(b) *Assume* $\mathbf{P}(X_0 = x > 0) = 1$. *Let* $\tau_0 = \inf\{t > 0;\ X_t = 0\}$. *Then*

$$\mathbf{P}\left(\int_0^{\tau_0} V(X_s)\,ds = +\infty \mid \tau_0 < +\infty\right) = 0 \ \text{ or } \ 1 \tag{5.3}$$

And the quantity in (5.3) *equals to one iff*

$$\int_0^1 V(r)\,r\,dr = +\infty \tag{5.4}$$

Proof. Remark at first that $\mathbf{P}(\tau_0 < +\infty) > 0$, (5.3) takes sense. We shall reduce to Brownian motion case by Girsanov formula and time change. At first, for every $t > 0$, define

$$\mathbf{Q} = \exp\left(-\int_0^t \frac{a_s}{\sigma_s^2}\,dM_s - \frac{1}{2}\int_0^t \left(\frac{a_s}{\sigma_s}\right)^2 ds\right) d\mathbf{P}$$

which is a probability measure equivalent to $\mathbf{P}$ on $\mathscr{F}_t$, by Novikov's criterion about exponential martingale and our conditions (i), (ii). $\mathbf{Q}$ is consistently defined on all $\mathscr{F}_t$, $t > 0$. Under $\mathbf{Q}$, $X = (X_t)$ becomes itself a continuous martingale with $\langle X\rangle = \langle M\rangle$, by Girsanov formula. Moreover $\mathbf{Q}$ is equivalent to $\mathbf{P}$ on $\mathscr{F}_\tau \cap [\tau < +\infty]$ for any stopping time τ, by [JS, Ch. IV] and our conditions (i) and (ii).

Second, under $\mathbf{Q}$, it is well known that $X_t = B_{\langle X\rangle_t}$ where $B_\cdot$ is a $\mathbf{Q}$-Brownian motion with respect to the time changed filtration $(\mathscr{F}_{T_t})$ with $T_t = \inf\{s;\ \langle X\rangle_s > t\}$ (see [RY, p. 170, Th. 1.6]). Notice $\mathbf{Q}$-*a.s.*

$$(1/C^2)\int_0^{\cdot/C^2} V(B_s)\,ds \leqslant \int_0^{\cdot} V(X_s)\,ds \leqslant C^2 \int_0^{C^2\cdot} V(B_s)\,ds$$

and

$$(1/C^2)\int_0^{\tau_0^B} V(B_s)\,ds \leqslant \int_0^{\tau_0} V(X_s)\,ds \leqslant C^2 \int_0^{\tau_0^B} V(B_s)\,ds$$

where $\tau_0^B = \inf\{t > 0;\ B_t = 0\}$.

Combining these discussions, we can assume without loss of generality $X = B$ is a BM w.r.t. $\mathbf{P}$.

(a) By Blumenthal 0-1 law, (5.1) holds for $X = B$. Let (L_t^r) be the local time at r until $t \in \mathbf{R}^+$. We have

$$\int_0^{\varepsilon} V(B_s)\,ds = \int_{-\infty}^{+\infty} V(r)\,L_\varepsilon^r\,dr$$

Since

$$\int_{[-1,1]^c} V(r)\, L_\varepsilon^r\, dr = \int_0^\varepsilon (1_{[-1,1]^c} V)(B_s)\, ds$$

which is finite **P**-*a.s.* by the condition that $1_{[-1,1]^c} V \in L^1_{loc}(\mathbf{R}, dr)$ (in fact this holds under $\mathbf{P}_x$ for quasi everywhere $x \in \mathbf{R}$ by Sturm [St], but quasi everywhere is the same as everywhere in the actual one dimension case), and since $\mathbf{P}(r \to L_\varepsilon^r$ is continuous and $L_\varepsilon^0 > 0) = 1$, we get immediately (5.2).

(b) We assume $x = 1$, *i.e.*, $\mathbf{P}(B_0 = 1) = 1$ for simplicity. In this case

$$\int_0^{\tau_0} 1_{[0,1]} V(B_s)\, ds = \int_0^1 V(r)\, L_{\tau_0}^r\, dr$$

The first key to the proof is the well-known Ray-Knight theorem ([RY, p. 421]: $(L_{\tau_0}^r)_{0 \leqslant r \leqslant 1}$ is a Bessel square process of dimension-2 starting from 0. In other words, let $(W_t)_{t \geqslant 0}$ be a 2-dimensional BM with $W_0 = 0$, then

$$(L_{\tau_0}^r)_{r \in [0,1]} \text{ equals in law with } (|W_t|^2)_{t \in [0,1]}$$

Thus we have the equality in law:

$$\int_0^1 V(r)\, L_{\tau_0}^r\, dr = \int_0^1 r V(r)(|W_r|^2/r)\, dr \tag{5.5}$$

By the Blumenthal 0-1 law (w.r.t. W), (5.3) holds.

The second ingredient is the Jeulin's Lemma (see [RY, the Hint in p. 425 and its historical notes in 434]): it claims that the r.h.s. of (5.5) is a.s. infinite iff (5.4) holds. ∎

Remarks (5.i). This lemma gives directly Corollary 1.6 for $X = B$, because (5.1) (resp. (5.3)) is exactly (C2) (resp. (C1)).

5.2. *Proof of Proposition* 1.5.

(a) Step 1. For every $z_0 \in \partial D$, and $r_0 > 0$ sufficiently small, as ∂D is C^3, and $\nabla \rho(z) = \mathbf{n}(z) \neq 0$, $\forall z \in \partial D$, where $\mathbf{n}(z)$ is the inner normal vector, $\rho(x) = dist(x, \partial D)$ is in $C_b^3(B(z_0, r_0) \cap \bar{D})$, and $|\nabla \rho| \geqslant 1/2 > 0$ on $B(z_0, r_0) \cap D$. Consider an extension $r(x) \in C_b^3(\mathbf{R}^d)$ such that:

(1) $r(x) = \rho(x)$, $\forall x \in B(z_0, r_0) \cap D$, and $r(x) \geqslant 0$, $\forall x \in \mathbf{R}^d$.

(2) $|\nabla r| \geqslant 1/2$ over $\mathbf{R}^d$;

Step 2. We prove now $r^2(B_t) = r^2(B_t^R)$, $\forall t < T := \inf\{t > 0;\ B_{(t)}^R \notin B(z_0, r_0)\}$ under $\mathbf{P}_\mu$ for any initial probability μ on $\bar{D}$. In fact, applying

Ito's formula and Skohorod's representation for the RBM: $dB_t^R = dB_t + \mathbf{n}(B_t^R)\, dL_t^{\partial D}$, where $L_\cdot^{\partial D}$ is the local time at ∂D (see Section 4.2), we have

$$r^2(B_t^R) - r^2(B_0^R) = \int_0^t 2r\nabla r(B_s^R)(dB_s + \mathbf{n}(B_s^R)\, dL_s^{\partial D}) + \tfrac{1}{2}\int_0^t (r^2)''\,(B_s^R)\, ds$$

$$= \int_0^t 2r\nabla r(B_s^R)\, dB_s + \tfrac{1}{2}\int_0^t (r^2)''\,(B_s^R)\, ds \tag{5.6}$$

where $r(B_\cdot^R)\, dL^{\partial D} = 0$ on $[0, T)$ is used. As $r \in C_b^3$, the coefficients in the above stochastic differential equation are C^1, then we have the unicity of its strong solution. Since $r^2(B_\cdot)$ satisfies (5.6) too, we get the desired claim.

Step 3. Observe that $X_\cdot = r(B_\cdot)$ is a semimartingale verifying all conditions in Lemma 5.1 by Ito's formula.

Notice that $\{r;\, \tilde V < a\} = \{r;\, \sigma_r(\Gamma_r \cap [V < a]) > 0\}$ for every real a, $\tilde V$ defined in Proposition 1.2 is Lebesgue measurable by Fubini's theorem. Since $\tilde V(\rho(x)) \leqslant V(x)$ on $B(z_0, r_0)$, dx-a.e. and the transition function $P_t^R(x, dy)$ of the RBM is absolutely continuous w.r.t. dy, we have $\mathbf{P}_{z_0}$-*a.s.*

$$\int_0^\varepsilon V(B_s^R)\, ds \geqslant \int_0^\varepsilon \tilde V(\rho(B_s^R))\, ds = \int_0^\varepsilon \tilde V(r(B_s))\, ds, \tag{5.7}$$

$\forall \varepsilon < T$ where the second equality follows from Step 2. By Lemma 5.1, the last term above is infinite $\mathbf{P}_{z_0}$-*a.s.*, hence (1.11) holds. By Theorem 1.4, (C.2) is satisfied.

(b) Under the auxiliary condition of (b), we have also the inverse of (5.7) below: $\mathbf{P}_z$-*a.s.* for every $z \in B(z_0, r_0) \cap \partial D$,

$$\int_0^\varepsilon V(B_s^R)\, ds \leqslant C\int_0^\varepsilon \tilde V(\rho(B_s^R))\, ds = C\int_0^\varepsilon \tilde V(r(B_s))\, ds, \qquad \forall \varepsilon < T. \tag{5.8}$$

If (C2) is satisfied, by Theorem 1.4(b.iii), there exists $z \in B(z_0, r_0) \cap \partial D$ such that

$$\mathbf{P}_z\left(\int_0^\varepsilon V(B_s^R)\, ds = +\infty,\ \forall 0 < \varepsilon < T\right) = 1.$$

Hence the last term in (5.8) is $\mathbf{P}_z$-*a.s.* infinite too. Now (b) follows from Lemma 5.1. ∎

5.3. *Proof of Proposition* 1.2

We begin by a general remark: for two nonnegative Borel measurable potentials V, V', if $V=V'$ on $B(z_0, r_0)$ where $z_0 \in \partial D$, $r_0 > 0$, then $\forall m$-*a.e.* $x_0 \in D$, we have $\mathbf{P}_{x_0}$-*a.s.* on $[B_{\tau_D} \in B(z_0, r_0)]$,

$$\int_0^{\tau_D} V(B_s)\, ds = +\infty \qquad \text{iff} \quad \int_0^{\tau_D} V'(B_s)\, ds = +\infty \tag{5.9}$$

(a) As ∂D is C^2, we construct an extension $r(x) \in C_b^2(\mathbf{R}^d)$ possessing all properties (1) and (2) in Step 1 of Proposition 1.5 above (except C^3). For every $z_0 \in \partial D$, $r_0 > 0$ small fixed, define $V' = 1_{B(z_0, r_0)} V$. We have $\mathbf{P}_{x_0}$-*a.s.*, (for m-*a.e.* $x_0 \in D$)

$$\left[\int_0^{\tau_D} \tilde{V}(r(B_s))\, ds = +\infty \right] \subset \left[\int_0^{\tau_D} V'(B_s)\, ds = +\infty \right] \tag{5.10}$$

on $[B_{\tau_D} \in B(z_0, r_0)]$, where $\tilde{V}(r)$ is given in Proposition 1.2.(a) for $r < r_0$, and $\tilde{V}(r) = 0$, $\forall r \geqslant r_0$. By Lemma 5.1 and our condition (1.4), the conditional probability of the l.h.s. of (5.10) knowing $[B_{\tau_D} \in B(z_0, r_0)]$ under $\mathbf{P}_{x_0}$ is one. As z_0, r_0 are arbitrary, hence the property in (5.9) holds with $\mathbf{P}_{x_0}$-probability one. This implies (C1).

(b) Under the assumption of (b), we have the inverse of (5.10), by which and (5.9), one can conclude easily from Lemma 5.1. ∎

6. APPLICATIONS: THE UNIQUE SOLVABILITY OF PDEs

In this section we present some applications of the L^1-e.gr. of S defined in Theorem 1.1.(i) to the heat diffusion equation, the resolvant equation and the eigenvalue problem of the Schrödinger operator.

6.1. *The Heat Diffusion Equation*

It is well known that the e.s.a. of S in $L^2(D)$ is equivalent to the unique solvability of the Schrödinger's equation, and to that of heat diffusion equation

$$\frac{\partial}{\partial t} u = (\Delta/2 - V)\, u \qquad \text{with} \qquad u(0, x) = f(x) \tag{6.1}$$

in $L^2(D)$. Let us introduce

Definition 6.1. A function $u(t, x)$ on $\mathbf{R}^+ \times D$ is called a $L^\infty(D)$-weak solution of (6.1) with the initial condition $u(0, x) = f(x) \in L^\infty(D)$, if

(i) for all $t \geqslant 0$, and all $\phi \in C_0^\infty(D)$,

$$\langle \phi, u(t, \cdot) - f \rangle = \int_0^t \langle u(s, \cdot), (\Delta/2 - V)\, \phi \rangle \, ds \tag{6.1b}$$

(ii) $u(t, \cdot) \in L^\infty(D)$ and $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^\infty(D)} < +\infty$ for all t, $T \geqslant 0$;

We adopt here the usual convention: one solution is in fact one equivalence class $\{\tilde{u} \mid \tilde{u}(t, \cdot) = u(t, \cdot),\ dx\text{-a.e.},\ \forall t \in \mathbf{R}^+\}$.

Notice that (i) means exactly that u is a generalized solution. Note also that for every strongly continuous semigroup (T_t) of bounded operators in $L^1(D)$, whose generator is an extension of S, $u(t, \cdot) := T_t^* f$ is a L^∞-weak solution of (6.1).

The signification of the L^1-e.gr. of S is clear from

THEOREM 6.2. *Assume* (H1). *If V satisfies* (C1), *then for every $f \in L^\infty(D)$, the heat diffusion equation* (6.1) *has one unique L^∞-weak solution, which is given by $u(t, x) = P_t^{D, V} f(x)$.*

Proof. Only the uniqueness requires proof. Let $u(t) = u(t, \cdot)$ be a L^∞-weak solution of (6.1). By (i) and (ii) in the definition, one can easily that $t \to u(t)$ is continuous from $\mathbf{R}^+$ to $(L^\infty, \sigma(L^\infty, L^1))$.

Fix $T > 0$ and $\phi \in C_0^\infty(D)$, set

$$h(t) := \langle \phi, P_t u(T - t) \rangle = \langle P_t \phi, u(T - t) \rangle.$$

where $P_t = P_t^{D, V}$ for simplicity of notation. Since $h(0) = \langle \phi, u(T) \rangle$, $h(T) = \langle \phi, P_T^{D, V} f \rangle$, for the unicity, it is enough to show that $h(0) = h(T)$.

For every $0 < t < T$, let us calculate

$$h'_+(t) := \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} (h(t + \varepsilon) - h(t)).$$

Write

$$\begin{aligned} h(t + \varepsilon) - h(t) &= \langle P_{t+\varepsilon} \phi, u(T - t - \varepsilon) \rangle - \langle P_t \phi, u(T - t) \rangle \\ &= \langle P_{t+\varepsilon} \phi - P_t \phi, u(T - t - \varepsilon) \rangle \\ &\quad + \langle P_t \phi, u(T - t - \varepsilon) - u(T - t) \rangle \\ &:= (I) + (II) \end{aligned}$$

By Lemma 2.3, as $\varepsilon \to 0+$,

$$\frac{1}{\varepsilon} (P_{t+\varepsilon} \phi - P_t \phi) \to -P_t S \phi \qquad \text{in } L^1(D), \tag{6.2}$$

which, combining with the fact that $u(T-t-\varepsilon) \to u(T-t)$ in $\sigma(L^\infty, L^1)$ noted previously, implies

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}(I) = -\langle P_t S\phi, u(T-t)\rangle \tag{6.3}$$

To trait (II), we require the key fact below:

$$\langle \psi, u(t)-f\rangle = -\int_0^t \langle u(s), A\psi\rangle \, ds, \qquad \forall \psi \in \mathscr{D}(A) \tag{6.4}$$

where $A = A_D^{(1)}$. To this end, (C1) implies that A is the closure of S in $L^1(D)$ by Theorem 1.1. Hence $\forall \psi \in \mathscr{D}(A)$, choose $\phi_n \in C_0^\infty(D)$, $n \in \mathbf{N}$, such that

$$\phi_n \to \psi, \qquad S\phi_n \to A\psi \qquad \text{in} \quad L^1(D).$$

We get (6.4) from (6.1b) by dominated convergence.

Since $t \to \langle A\psi, u(t)\rangle$ is continuous on $\mathbf{R}^+$ by the continuity of $t \to u(t)$ w.r.t. $\sigma(L^\infty, L^1)$, (6.4) says that $\langle \psi, u(t)\rangle$ is continuously differentiable and

$$\frac{d}{dt}\langle \psi, u(t)\rangle = -\langle u(t), A\psi\rangle.$$

Applying this property to $\psi = P_t\phi \in \mathscr{D}(A)$ to (II), we get

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}(II) = \langle AP_t\phi, u(T-t)\rangle \tag{6.5}$$

But as $AP_t\phi = P_tA\phi = P_tS\phi$, we get by (6.3) and (6.5) that

$$h'_+(t) = 0, \qquad \forall t \in (0, T).$$

Since $h(t)$ is obviously continuous on $[0, T]$, the fact above implies by the classical Dini's theorem that h is constant on $[0, T]$, the desired result. ∎

Remarks (6.i). The e.s.a. of S is equivalent to the uniqueness of solution u of (6.1) satisfying (6.1b) and

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_2 < +\infty, \qquad \forall T > 0.$$

(6.ii). We have not been able to show that (C1) is equivalent to the uniqueness of the C_0-semigroup in L^1 whose generator extends S. If that were the case, (C1) is also necessary to the unique L^∞-weak solvability of (6.1).

(6.iii). See Lemma 6.3 below for the good properties of $P_t^{D,V}f$.

6.2. *The Resolvant Equation and the Eigenvalue Problem*

We denote by $C_\partial(D)$ the Banach space of continuous functions tending to zero as $x \in D$ approaches to the one point ∂ compactification of D, equipped with the sup norm. The same for $C_\partial(D \times D)$.

LEMMA 6.3. *Assume* (H1),

$$V^+ \in \mathcal{K}_{loc}(D), m(D) < \infty \qquad \text{and} \qquad D \text{ is connected and regular} \tag{6.6}$$

(i.e., $\forall z \in \partial D$, $\mathbf{P}_z(\tau_D > 0) = 0$).

(a) *Then there is* $p_t^{D,V}(x, y) \in C_\partial(D \times D)$ *strictly positive and symmetric on* $D \times D$ *such that*

$$P_t^{D,V} f(x) = \int_D f(y)\, p_t^{D,V}(x, y)\, dy$$

for every $f \geqslant 0$ *measurable. And*

$$P_t^{D,V} f \in C_\partial(D) \text{ for any } f \in L^p(D),\ p \in [1, +\infty] \tag{6.7}$$

$(P_t^{D,V})$ *is a strongly continuous semigroup of bounded operators on* $C_\partial(D)$.

Moreover $\forall t > 0$, $P_t^{D,V}$ *is of trace class on* $L^2(D)$. *Let* $(\phi_k)_{k \in \mathbf{N}}$ *be the ONB composed of eigenfunctions associated to the eigenvalues* $(e^{-\lambda_k t}; k \in \mathbf{N})$ *where* $\lambda_k \to +\infty$, $\sum_k \exp(-\lambda_k t) < +\infty$, *then* $\phi_k \in C_\partial(D)$ *and*

$$p_t^{D,V}(x, y) = \sum_{k=0}^{\infty} \exp(-\lambda_k t)\, \phi_k(x)\, \phi_k(y) \tag{6.8}$$

where the serie is absolutely convergent in $C_\partial(D \times D)$. *In particular* $P_t^{D,V}$ *is compact and has the same eigenvalues and the same eigenfunctions in each of* $X = L^p$, $p \in [1, +\infty]$ *or* $X = C_\partial(D)$.

(b) *Let* A_D^X *be the generator of* $(P_t^{D,V})$ *on* $X = L^p(D)$, $p \in [1, +\infty)$ *or* $X = C_\partial(D)$. *Then the spectrum of* A_D^X *in* X *is* $\Lambda^{D,V} = \{\lambda_k; k \in \mathbf{N}\}$, *the spectrum of* A_D *in* $L^2(D)$, *and* $\forall \lambda \in \Lambda^{D,V}$, *its eigensubspace in* X *is generated by those* ϕ_k, *the eigenfunctions associated to* λ *in* L^2.

Proof. (a) This is known when $V \in \mathcal{K}$ (see [CZ, Th. 3.10; 3.17]). In the general situation, take one sequence of relative compact open regular (D_n) such that $\bar{D}_n \subset D_{n+1} \uparrow D$. For each n, take $V = V^+ 1_{D_n} - V^- \in \mathcal{K}$, we have for every $f \geqslant 0$ measurable,

$$\mathbf{E}^x f(B_t)\, 1_{[t < \tau_{D_n}]} \exp -\int_0^t V(B_s)\, ds \leqslant P_t^{D,V} f(x) \leqslant P_t^{D,V_n} f(x) \tag{6.9}$$

Write $T_t^n f(x)$, $\tilde{T}_t^n f(x)$ for the left and right hand sides of (6.9).

Let us remark that the boundedness of $P_t^{D,V}$ from $L^1(D)$ to $L^\infty(D)$, implies: there is a bounded Borel function $P_t^{D,V}$ on $D \times D$ such that

$$P_t^{D,V}(x, dy) = p_t^{D,V}(x, y)\, dy \qquad \text{and} \qquad p_t^{D,V} \in L^\infty(D \times D)$$

see Simon [Si, (A1.2)]. Hence $P_t^{D,V}$ belongs to the Hilbert–Schmidt class (as $m(D) < +\infty$) for each $t > 0$. Therefore $P_t^{D,V} = P_{t/2}^{D,V} P_{t/2}^{D,V}$ is of trace class.

To show (6.7), since $P_{t/2}^{D,V} f \in L^\infty$ and $P_t^{D,V} f = P_{t/2}^{D,V}(P_{t/2}^{D,V} f)$, we can assume that $f \in L^\infty$. Fix a compact subset $\mathbf{K}$ of D, we show now

$$\sup_{x \in \mathbf{K}} |P_t^{D,V} f(x) - T_t^n f(x)| \to 0, \qquad \text{as} \quad n \to \infty \tag{6.10}$$

To this end, $\forall x \in K \subset D_n$ (n large),

$$\begin{aligned} 0 &\leqslant P_t^{D,V} f(x) - T_t^n f(x) = \mathbf{E}^x f(B_t)\, 1_{[\tau_{D_n} \leqslant t < \tau_D]} \exp - \int_0^t V(B_s)\, ds \\ &\leqslant [\mathbf{P}_x(\tau_{D_n} \leqslant t < \tau_D)\, P_t^{D,-2V^-} f^2(x)]^{1/2} \\ &\leqslant C \sqrt{\mathbf{P}_x(\tau_{D_n} \leqslant t < \tau_D)} \end{aligned}$$

where

$$C = \sup_{x \in D} |P_t^{D,2V^-} f^2(x)| < +\infty.$$

By an elementary argument (taking small ball), we can prove that $x \to \mathbf{P}_x(\tau_{D_n} \leqslant t < \tau_D)$ is continuous on $\mathbf{K}$. As n increases to infinity, this sequence of continuous functions decreases to zero. Hence Dini's monotone convergence theorem implies that convergence is uniform on $\mathbf{K}$. So (6.10) follows. As $\mathbf{K}$ is arbitrary, $P_t^{D,V} f$ is continuous on D. Moreover, by [CZ, Prop. 3.14],

$$\lim_{x \to \partial} |P_t^{D,V} f(x)| \leqslant \lim_{x \to \partial} P_t^{D,V_n} |f|\,(x) = 0.$$

Then (6.7) follows.

To the second claim in (a), its spectral structure in $L^2(D)$ presented here follows from the symmetry and the trace property of $P_t^{D,V}$, and the spectral decomposition in L^2. That $\phi_k \in C_\partial(D)$ follows from

$$\exp(-\lambda_k t)\, \phi_k = P_t^{D,V} \phi_k \in C_\partial(D)$$

by (6.7). The above formula gives also

$$\|\phi_k\|_\infty \leqslant \exp(\lambda_k s)\, \|P_s^{D,V}\|_{2,\infty}, \qquad \forall s > 0, \quad k \in \mathbf{N}$$

Applying it with $0<s<t/2$, we have by the trace property of $P_{t-2s}^{D,V}$,

$$\sum_{k=0}^{\infty} \exp(-\lambda_k t) \sup_{s,\, y\in D} |\phi_k(x)\,\phi_k(y)|$$

$$\leqslant \sum_{k=0}^{\infty} \exp(-\lambda_k(t-2s))\, \|P_s^{D,V}\|_{2,\infty} < +\infty,$$

This is absolute convergence in (6.8). In particular $P_t^{D,V} \in C_\partial(D\times D)$. Since T_t^n defined in the l.h.s. of (6.9) has a strictly positive density $p_t^n(x,y)$ on $D_n \times D_n$ by [CZ], which must be smaller than $p_t^{D,V}$ by (6.9), this last is then strictly positive on D.

For the last claim, let

$$P_t^n f := \sum_{k=0}^{n} \exp(-\lambda_k t)\, \phi_k \langle \phi_k, f\rangle = P_t^{D,V} \Pi_n f$$

where

$$\Pi_n f = \sum_{k=0}^{n} \phi_k \,\langle \phi_k, f\rangle.$$

The uniform convergence in (6.8) implies that: as $n\to\infty$,

$$\sup\{\|P_t^n f - P_t^{D,V} f\|_{C_\partial(D)};\ \|f\|_1 \leqslant 1\} \to 0.$$

Fix $X=L^p(D)$, $p\in[1,\infty]$ or $X=C_\partial(D)$. The above convergence implies that P_t^n converges in operator norm to $P_t^{D,V}$ on X, then $P_t^{D,V}$ is compact. It is easy to see that $\lambda=0$ is not an eigenvalue of $P_t^{D,V}$ in X (but 0 is a point in the spectrum of $P_t^{D,V}$ in X).

Finally let (λ, f) with $\lambda\neq 0$ be a couple of eigenvalue and eigenfunction of $P_t^{D,V}$ in X, by the boundedness of $P_t^{D,V}\colon X\to L^2$, we see that $f\in L^2$. Hence (λ, f) is a such couple in L^2 too. The inverse is true also, because $\phi_k \in C_\partial(D)\subset X$.

(b) Take $\alpha > -\lambda_0$, we have

$$\int_2^\infty e^{-\alpha t}\, \|P_t^{D,V}\|_X\, dt$$

$$\leqslant \int_2^\infty e^{-\alpha t}\, \|P_1^{D,V}\|_{X\to L^2}\cdot \|P_{t-2}^{D,V}\|_2 \cdot \|P_1^{D,V}\|_{L^2\to X}\, dt < +\infty$$

$$\text{and} \qquad \sup_{t\leqslant 2} \|P_t^{D,V}\|_X < +\infty. \tag{6.11}$$

(the last follows from (2.2), duality and interpolation). This implies that α belongs to the resolvant set of the generator $-A^X$ of $(P_t^{D,V})$ on X and

$$R_\alpha^X = (\alpha + A^X)^{-1} = \int_0^\infty e^{-\alpha t} P_t^{D,V}\, dt$$

is compact on X. In the following we pass to the complexication of X, denoted still by X.

By Kato [Th. 6.29], the spectrum of A^X is composed only of the eigenvalues $\{\mu_k \in \mathbf{C};\ |\mu_k| \to +\infty\}$. But for $f \in X$, $\lambda \in \mathbf{C}$

$$A^X f = \lambda f \Leftrightarrow P_t^{D,V} f = e^{-\lambda t} f, \qquad \forall t > 0. \tag{6.12}$$

By the spectral structure of $P_t^{D,V}$ in X is given in (a), this leads to the desired result. ∎

THEOREM 6.4. *Assume* (H1) *and* (6.6).

(a) *Let* $\lambda \notin \Lambda^{D,V}\{\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots\}$, *the spectrum of the Friedrichs extension* A_D *of* S *in* $L^2(D)$. *For each* $f \in L^\infty(D)$, *the resolvant equation*

$$-(\lambda + \Delta/2 - V)\, u = f \tag{6.13}$$

has one unique solution $u \in L^\infty(D)$ *in the distribution sense if and only if* V *satisfies* (C1). *In this case, if* $\lambda < \lambda_0$, *the unique solution is given by*

$$u(x) = \int_0^\infty e^{\lambda t} P_t^{D,V} f(x)\, dt \tag{6.14}$$

which is a continuous function belonging to $C_\partial(D)$.

(b) *Assume* (C1). *Let* $\lambda \in \Lambda^{D,V}$. *Any solution* $u \in L^\infty(D)$ *in the distribution sense of*

$$(\lambda + \Delta/2 - V)\, u = 0 \tag{6.15}$$

lies in the finite dimensional space generated by the eigenfunctions $\phi_k \in C_\partial(D)$ *of* A_D *associated to* λ.

Proof. We begin by a remark. Let $S_\infty^*: \mathscr{D}(S_\infty^*) \subset L^\infty \to L^\infty$ be the dual operator of S: $C_0^\infty(D) \subset L^1 \to L^1$. It is easy to check that

$$\begin{aligned} \mathscr{D}(S_\infty^*) &= \{f \in L^\infty(D)\colon (-\Delta/2 + V)\, f \in L^\infty(D)\} \\ S_\infty^* f &= (-\Delta/2 + V)\, f, \qquad \forall f \in \mathscr{D}(S_\infty^*) \end{aligned} \tag{6.16}$$

(a) *Necessity.* If (6.13) has one unique solution in L^∞, so does the homogeneous equation (6.13) with $f=0$. But this last property can be interpreted as

$$(\lambda - S_\infty^*)\, u = 0 \Rightarrow u = 0$$

which is equivalent to the fact that $(\lambda - S)(C_0^\infty(D))$ is dense in L^1. By Lemma 2.6 (iii) $\Rightarrow$ (i) (as $\lambda \in \rho(A_D^{(1)})$ by Lemma 6.3), the closure of S in $L^1(D)$ is $A_D^{(1)}$. Hence (C1) is satisfied by Theorem 1.1.

Sufficiency. If (C1) is valid, then $A_D^{(1)}$ is the closure of S in $L^1(S)$. Then $(A_D^{(1)})^* = S_\infty^*$. By Lemma 6.3(b), the spectrum of $A_D^{(1)}$ is $\Lambda^{D,V}$. Thus for $\lambda \notin \Lambda^{D,V}$, $\bar\lambda \notin \Lambda^{D,V}$, the resolvant

$$R_{\bar\lambda}^{(1)} := (\bar\lambda - A_D^{(1)})^{-1}$$

is bounded on $L^1(D)$, where $\bar\lambda$ is the conjugate of $\lambda \in \mathbf{C}$. Thus λ belongs to the resolvant set of S_∞^* and

$$(\lambda - S_\infty^*)^{-1} \colon L^\infty(D) \to L^\infty(D)$$

is bounded and equals to $(R_{\bar\lambda}^{(1)})^*$ ([Ka, Th. 6.22, p. 184]). As (6.13) for $u \in L^\infty$ is equivalent to

$$-(\lambda - S_\infty^*)\, u = f,$$

thus $u = -(\lambda - S_\infty^*)^{-1} f$ is the unique solution in L^∞.

If $\lambda < \lambda_0$ (the smallest of $\Lambda^{D,V}$),

$$-R_\lambda^{(1)} = \int_0^\infty e^{\lambda t} P_t^{D,V}\, dt$$

and the last integral is absolutely convergent in the operator norm in $L^1(D)$ by (6.11). Since its dual operator is $-(\lambda - S_\infty^*)^{-1}$ (shown above), (6.14) holds.

Finally $\forall \varepsilon > 0$, $P_\varepsilon^{D,V} f \in C_\partial(D)$, we get by (6.11) with $X = C_\partial(D)$,

$$\int_\varepsilon^\infty e^{\lambda t} P_t^{D,V} f\, dt = \int_0^\infty e^{\lambda t} P_t^{D,V} P_\varepsilon f\, dt \in C_\partial(D).$$

But as $V^- \in \mathscr{K}$, we have also

$$\int_0^\varepsilon e^{\lambda t} \sup_{x \in D} |P_t^{D,V} f(x)|\, dt \leqslant \frac{\varepsilon}{\lambda} (e^{\varepsilon\lambda} - 1) \sup_{x \in D,\, t \in [0,\varepsilon]} |P_t^{-V^-} f(x)|$$

which tends to zero as $\varepsilon \to 0$ by (2.2). The last claim follows.

(b) The equation (6.15) is equivalent to

$$S^*_\infty u = \lambda u. \tag{6.17}$$

By the key fact that $S^*_\infty = (A_D^{(1)})^*$ noted in the proof of (a), (6.17) is equivalent to

$$\langle u, (\lambda - A_D^{(1)})\, f \rangle = 0, \qquad \forall f \in \mathscr{D}(A_D^{(1)}).$$

Take $f = \phi_m$, the eigenfunction of $A_D^{(1)}$ associated to $\lambda_m \neq \lambda$ specified in Lemma 6.3(c). We have $\langle u, \phi_m \rangle = 0$. As $u \in L^\infty \subset L^2$, we get the desired result. ▌

Remarks (iv). The assumption (6.6) is fairly general. The most restrictive in it is perhaps $m(D) < +\infty$. If we substitute it by

$$\lim_{|x| \to \infty,\, x \in D} V^+(x) = \infty \tag{6.17}$$

we have still the compactness of $P_t^{D, V}$ in $L^2(D)$. By interpolation, it will be compact in all $L^p(D)$, $p \in (1, +\infty)$. But I do not know the limit case $p = 1$ or $+\infty$. It is a challenge question to regard Lemma 6.3, Theorem 6.4 in this situation.

(6.v) In the classical context that $V \in \mathscr{K}$ in (6.6), it is well known (see [AS] and [CZ] e.g.) that the resolvant equation (6.13) has one unique solution $u \in C_b(\bar{D})$ satisfying the boundary $u|_{\partial D} = g \in C_b(\partial D)$ for any $\lambda \in \mathbf{R}$ (then it has an infinite number of solutions without the boundary condition). And the equation (6.15) has one unique solution $u \in C_b(\bar{D})$ satisfying the Dirichlet boundary condition $u|_{\partial D} = g$, where $g \in C_b(\partial D)$, $\lambda \in \mathbf{R}$ are arbitrary.

But under (C1), the situation is completely different:

(1) (6.13) has not require the boundary condition that $u|_{\partial D} = 0$, because it is automatically verified in both situations (a) and (b).

(2) the Dirichlet problem associated to (6.15) has nonzero solution only if $g = 0$ on ∂D and $\lambda \in \Lambda^{D, V}$ (by part (b)).

In other words, the boundary effect is eliminated by (C1), justifying the probabilistic intuitive picture described at the beginning of the paper.

(6.vi) Assume (H1), (6.6), (C1), and $V \in L^2_{loc}(D)$. Let $\mathscr{A}_c(S)$ be the family of the self-adjoint extensions A of S in L^2 such that $(i + A)^{-1}$ is compact. By the s.a. extension theory [RS, Th. X.2 and its Corollary], the e.s.a. is equivalent to the fact that $\mathscr{A}_c(S)$ is a singleton. By Theorem 6.4, if $A \in \mathscr{A}_c(S)$ has only $L^\infty(D)$-eigenfunctions ϕ, then $A = A_D$ (left to the reader). In the Schrödinger equation's interpretation, this means that if the quantum system has only bounded excited states (eigenfunctions in Mathematics), then it is completely determined by A_D.

Note in one-dimensional case (i.e., D is an interval), $\mathscr{A}(S)=\mathscr{A}_c(S)$ (see [RS, p. 146–161]).

Note finally an eigenfunction ϕ in L^2 of $A\in\mathscr{A}(S)$ (satisfies (6.15), is continuous by a remarkable result of Aizenman–Simon [AS] (see also [CZ, Th.5.21]). Then that $\phi\in L^\infty(D)$ is only a requirement near ∂D.

7. SEVERAL EXAMPLES

7.1.

We shall construct a counter-example to show that (1.3) (resp. (1.10)) does not imply (C1) (resp. (C2)) in the high dimensional situation ($d\geqslant 2$), unlike the one dimension case in Corollary 1.6.

We shall take $D=(0,+\infty)\times\mathbf{R}$, the demi-plan in $\mathbf{R}^2$. Let $\mathbf{T}=\{x_{n,k}:=(1/2^n, k/2^n);\ n\in\mathbf{N},\ k\in\mathbf{Z}\}$. For each n, k, we take a closed ball $B_{n,k}$ centered at $x_{n,k}$ with rayon $r_n\leqslant 1/2^{2(n+1)}$ such that

$$Cap^D(B_{n,k})\leqslant\frac{1}{2^{2(n+1)}}\tag{7.1}$$

where Cap^D is the (1,2)-capacity w.r.t. $\mathscr{E}_D$ (or the killed BM). Remark that this family of balls are disjoints, and as $N\to\infty$,

$$Cap^D\left(B(z,1)\cap\bigcup_{n\geqslant N,\,k\in\mathbf{Z}}B_{n,k}\right)\leqslant\sum_{n=N}^{+\infty}\frac{2^n}{2^{2n}}=\frac{1}{2^{N-1}}\to 0\tag{7.2}$$

for every $z=(0,x_2)\in\partial D$. Notice that (7.2) holds when Cap^D is substituted by the (1, 2)-capacity of the BM or the RBM (because Cap^D is the largest among these three capacities).

Write $C=\bigcup_{n,k}B_{n,k}$. We have

(i) $D\backslash C$ is relatively open in D (but not open in $\mathbf{R}^2$ because of its behavior near ∂D);

(ii) $\bar{D}\backslash C$ is finely open w.r.t. the RBM $B^R=(|B^1|, B^2)$, where $B=(B^1,B^2)$ is the 2-dimensional BM (this property follows from (7.2) w.r.t. B^R).

A counter-example to (1.10). Now choose a nonnegative potential $V\colon\mathbf{R}^2\to\mathbf{R}^+$ such that

$$(1) \quad V \in C^\infty(D) \qquad \text{and} \qquad V = 0 \qquad \text{on } \mathbf{R}^2 \backslash C;$$

$$(2) \quad \int_{B_{n,k}} V\,dx = 1, \qquad \forall n \in \mathbf{N}, \quad k \in \mathbf{Z}.$$

Obviously (1.10) is satisfied, as every ball $B(z, r)$ with $z \in \partial D$ contains an infinite number of balls in C. But the property (b.v) in Theorem 1.4 is not satisfied by $O = \bar{D} \backslash C$, because $V = 0$ on $\bar{D} \backslash C$. In other words, V satisfies (1.10), but not (C2).

Moreover, by Theorem 1.4(b.ii), $\mathscr{E}_R^V$ is not regular (for the much more general construction of non-regular Dirichlet form, see [AM2]).

A counter-example to (1.3). Choose a nonnegative V on $\mathbf{R}^2$ satisfying (1) above and

$$(2') \qquad \int_{B_{n,k}} V(x_1, x_2)\, x_2\, dx = 1.$$

(instead of 2) above). It is clear that V satisfies (1.3). But V does not verify (C1), which follows from Theorem 1.1(iv) and the fact that $V = 0$ on $O = D \backslash C$, which is charged by the BM until τ_D by (7.2).

Remarks. Take $\tilde{V}(x_1, x_2) = V + 1\sqrt{x_2}$, where V is constructed above. Then $\tilde{V}$ satisfies still (1.3) or (1.10), but does not verify (C1) or (C2). This example is more stringent, because $\tilde{V}(x_1, x_2) \to +\infty$ as $x_2 \to 0+$.

7.2.

We present several simple examples to illustrate differences of the three notions: L^1-e.gr., e.m.s.a. and e.s.a.

EXAMPLE 1. Let $V = 0$ on D. We have

(a) S is L^1-e.gr. iff S is e.m.s.a., iff $Cap(D^c) = 0$ or $N = D^c$ is polar.

(b) S is e.s.a. iff D^c is of zero capacity (2.2).

In particular for $D = \mathbf{R}^d \backslash \{0\}$, S is L^1-e.gr. or e.m.s.a. iff $d \leqslant 2$, but S is e.s.a. iff $d \geqslant 4$.

EXAMPLE 2. Let $D = \mathbf{R}^n \times (0, +\infty)$ and $0 < V \in C^1(D)$ such that

$$\lim_{x_{n+1} \to 0+} V(y, x_{n+1}) / V(y', x_{n+1}) \in L^\infty_{loc}(\mathbf{R}^n \times \mathbf{R}^n).$$

We have by Proposition 1.2 and 1.5.

(a) S is L^1-e.gr. iff $\int_{B(z,r)} x_{n+1} V(x)dx = \infty$, $\forall z = (x_1, ..., x_n, 0) \in \partial D$, $r > 0$.

(b) S is e.m.s.a. iff $\int_{B(z,r)} V(x)\,dx = \infty$, $\forall z = (x_1, ..., x_n, 0) \in \partial D$, $r > 0$.

In particular for $V(x) = \beta / x_{n+1}^{\alpha}$, where $\alpha \in \mathbf{R}$, $\beta > 0$, S is L^1-e.gr. iff $\alpha \geqslant 2$, it is e.m.s.a. iff $\alpha \geqslant 1$. From the one-dimensional case, one can prove easily that S is e.s.a. iff ($\alpha = 2$ and $\beta \geqslant 3/8$) or $\alpha > 2$.

Example 3. Let $D = B(0, 1)$, the unit ball in $\mathbf{R}^d$, and $0 \leqslant V \in C^1(D)$. Assume $\exists C > 0$ such that

$$V(x) \leqslant CV(y), \qquad \forall |x| = |y| = r \in (0, 1).$$

By Proposition 1.2, and 1.5, we have

(a) S is L^1-e.gr. iff $\int_{B(z,\varepsilon)} (1 - |x|)\, V(x)\,dx = \infty$, $\forall z \in \partial D = S^d$, $\varepsilon > 0$.

(b) S is e.m.s.a. iff $\int_{B(z,r)} V(x)\,dx = \infty$, $\forall z \in \partial D$, $r > 0$.

ACKNOWLEDGMENTS

I am grateful to Professor A. B. Crouzeiro and J. C. Zembrini for their kind invitation to the University of Lisbon and to the Conference on Mathematical Physics held March 1997 at Madera, where this work was reported. This work is partially supported by the NSF of China and the Y. D. Fok's Foundation.

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